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Hiromi Murakami and Ryo-Ichi Nagahisa

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関西大学ソシオネットワーク戦略研究機構

Research Institute for Socionetwork Strategies,
Kansai University
Joint Usage / Research Center, MEXT, Japan
Suita, Osaka, 564-8680, Japan
URL: https://www.kansai-u.ac.jp/riss/index.html
e-mail: riss@ml.kandai.jp
tel. 06-6368-1228
fax. 06-6330-3304

On the resource allocation problems with interpersonal comparisons of welfare: Extended preference approach revisited*

Hiromi Murakami† and Ryo-Ichi Nagahisa‡

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Abstract

This study incorporates the extended preference approach—originally developed in the literature on social welfare functionals—into a pure exchange economy model. It analyzes the relationship between equal-income Walrasian allocations and leximin-equitable allocations and demonstrates that two axioms based on the leximin criterion characterize the EIW rule.

Keywords: equal-income Walras rule; extended preference; leximin criterion; interpersonal comparisons of utility levels

1 Introduction

This study applies the extended preference approach—a framework widely used in the theory of social welfare functionals (SWFLs)—to the resource allocation problem in exchange economies with finitely many agents and goods.[1]

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†Faculty of Fine Arts, Kyoto City University of Arts, 57-1 Shimono-cho, Shimogyo-ku, Kyoto 600-8601, Japan Email: h_murakami@kcua.ac.jp

‡Department of Economics, Kansai University 3-3-35 Yamatecho Suita Osaka 564-8680 Japan E-mail: t940074@kansai-u.ac.jp

[1] Most earlier studies analyze SWFLs within the Arrovian framework. A few exceptions—Sen (1974a, 1974b) and Deschamps and Gevers (1978)—apply the theory to income distributions in single-commodity economies.

Surveys of the literature include d'Aspremont (1985), d'Aspremont and Gevers (2002), Bossert and Weymark (2004), Sen (1986), and Suzumura (1983), all encompassing many related issues such as the axiomatic characterization of rules based on this approach and

An extended preference allows for interpersonal welfare comparisons; in particular, it enables us to assess whether being agent i with consumption x is at least as well off as being agent j with consumption y. Using this concept, we examine leximin fairness, a lexicographic refinement of Rawls's (1971) difference principle. A feasible allocation is leximin-equitable if no other feasible allocation is strictly superior from the leximin perspective. We impose three properties (E.1–E.3) on extended preferences, which play a central role in determining the existence and characteristics of leximin-equitable allocations (Theorems 1 and 2).

We then study the relationship between leximin-equitable allocations and equal-income Walrasian (EIW) allocations. Theorem 3 shows that any two of the following three notions jointly imply the third: leximin-equitable allocations, EIW allocations, and price-indicator-type extended preferences. The price-indicator type refers to a class of extended preferences in which agents' welfare levels are assessed using given price vectors.

The ultimate objective of the study is to axiomatize the EIW rule using the leximin criterion. A rule determines a social outcome by employing leximin assessments derived from extended preferences. We assume that such a rule adheres to the normative rationale of leximin fairness: the rule selects allocations on grounds that are meaningfully connected to the leximin criterion. To formalize this idea, we introduce Leximin Justification Possibility (LMJP) and Leximin All Unanimity (LMAU). For any feasible allocation, LMJP requires that if the rule selects it, then there must exist at least one extended preference under which the allocation becomes leximin-equitable. LMAU requires that if

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informational restrictions represented by invariance axioms. Fleurbaey and Hammond (2004) and Mongin and d'Aspremont (2004) are also surveys that take a broader perspective on the field, incorporating ethical and utility-theoretic considerations.

Blackorby et al. (1984) provide a diagrammatic introduction.

Hammond (1991) provides a comprehensive survey of interpersonal comparisons of utilities, including the extended sympathy approach.

such a justification holds for all extended preferences, then the rule must select the allocation. We show that the EIW rule is the unique rule that satisfies both axioms (Theorems 4 and 5).

The price-indicator type is logically equivalent to the notion of a locally created extended preference, which relies solely on local information about subjective preferences and incorporates the idea of local independence introduced by Nagahisa (1991) and Nagahisa and Suh (1995). This implies that introducing the price-indicator type plays the same role as relaxing the IIA condition in Arrow's framework.

The study uses the utility representation of extended preferences (Proposition 1) solely as a tool for interpersonal comparisons of utility levels (ICUL), not for interpersonal comparisons of utility differences (ICUD). In this sense, the representation is merely a matter of convenience, much like the standard use of utility functions in microeconomic analysis.

The remainder of the paper is organized as follows. Section 2 introduces notation and definitions. Section 3 investigates the properties of leximin-equitable allocations. Section 4 develops the notion of price-indicator-type extended preferences and analyzes their relationship to EIW allocations. Section 5 presents the axiomatization of the EIW rule. Section 6 defines locally created extended preferences and establishes their equivalence to price-indicator types. Section 7 addresses several issues related to previous studies. Section 8 concludes. The appendix contains additional technical material.

2 Notation and definitions

2.1 Exchange Economies

We use the notation $\geq$, $>$, and $\gg$ for vector inequalities. Define $\Delta^l := \{p \in R^l_+ : \sum_{i=1}^{l} p_i = 1\}$ and $int\Delta^l := \{p \in R^l_{++} : \sum_{i=1}^{l} p_i = 1\}$.

Consider exchange economies with a finite number of agents and a finite number of private goods. Let $N = \{1, 2, ..., n\}$ and $L = \{1, 2, ..., l\}$ be the set of agents and the set of private goods, respectively. Each agent has the same consumption set R^l_+. Let $z_i = (z_{i1}, ..., z_{il}) \in R^l_+$ denote agent i's consumption, and let $z = (z_1, ..., z_n) \in R^{nl}_+$ denote an allocation. Let $\Omega \in R^l_{++}$ represent the total endowment of the economy, which is collectively owned and fixed throughout the analysis. An allocation z is feasible if $\sum_{i \in N} z_i \leq \Omega$. Let Z denote the set of all feasible allocations.

Let $\succcurlyeq_i$ be agent i's preference over R^l_+, with associated strict preference $\succ_i$ and indifference $\sim_i$ defined in the usual way. We assume that $\succcurlyeq_i$ is continuous, convex, and monotonic on R^l_+. Specifically, $\succcurlyeq_i$ is convex if for any $x, y \in R^l_+$ with $x \succ_i y$, it follows that $tx + (1-t)y \succ_i y$ for all $t \in (0, 1)$. It is monotonic if $x > y$ implies $x \succ_i y$. Let Q be the set of all preferences satisfying these conditions. A list of all agents' preferences, denoted $\succcurlyeq = (\succcurlyeq_i)_{i \in N}$, is called a profile. The set of all such profiles is denoted $Q^n = Q \times \cdots \times Q$ (n times).

Given a profile $\succcurlyeq$, a feasible allocation z is said to be Pareto optimal if and only if there is no other feasible allocation z' with $z'_i \succ_i z_i$ for all $i \in N$.[2] Let $PO(\succcurlyeq)$ be the set of Pareto optimal allocations. Let $PO(\succcurlyeq, p)$ denote the set of Pareto optimal allocations supported by the price vector p. A feasible allocation z is said to be an equal income Walrasian (EIW) allocation if there exists some $p \in int\Delta^l$ such that, for all $x \in R^l_+$, $px \leq p\left(\frac{\Omega}{n}\right)$ implies $z_i \succcurlyeq_i x$. Let $EIW(\succcurlyeq)$ be the set of all such allocations. Let $EIW(\succcurlyeq, p)$ be the subset of $EIW(\succcurlyeq)$ supported by the price vector $p \in int\Delta^l$.

2.2 Extended Preferences

The notion of extended preferences builds on the idea of extended sympathy, introduced by Arrow (1963) and further developed by Suppes (1966) and Sen

[2] Because preferences are continuous and monotonic on R^l_+, strong and weak Pareto optimality coincide.

(1970). The underlying intuition is that a hypothetical ethical observer evaluates individuals' welfare from a social vantage point, taking into account—i.e., sympathizing with—their subjective preferences.

An extended preference, denoted by $\succsim_E$ and generated from a profile $\succsim$ in Q^n, is defined as a complete and transitive binary relation on $R^l_+ \times N$. The expression $(x, i) \succsim_E (y, j)$ means that "being agent i with consumption x is at least as well off as being agent j with consumption y." The associated strict and indifference preferences, $\succ_E$ and $\sim_E$, are defined in the usual way.

A family of continuous utility functions $(u_i)_{i \in N}$ represents $\succsim_E$ if $(x, i) \succsim_E (y, j) \iff u_i(x) \geq u_j(y)$ for all $i, j \in N$ and all $x, y \in R^l_+$. We call the tuple $(u_i)_{i \in N}$ a *representation* of $\succsim_E$. We assume that $\succsim_E$ satisfies the following properties.

E.1. For any $i \in N$, $x \succsim_i y \iff (x, i) \succsim_E (y, i)$ for all $x, y \in R^l_+$.

E.2. $\succsim_E$ has a representation.

E.3. There exists $z^* \in Z$ such that $z^*_k \neq 0$ for all k, and $(z^*_i, i) \sim_E (z^*_j, j)$ for all i, j.

E.1, known as *the axiom of identity*, is standard in the literature on extended preferences.[3] E.2 ensures the existence of continuous utility representations and thereby allows—following much of the SWFL literature—utility information to serve as the basis for social evaluation. Only utility levels matter for these comparisons; utility differences do not. Accordingly, we adopt interpersonal comparisons of utility levels (ICUL) rather than interpersonal comparisons of utility differences (ICUD).

Determining whether $\succsim_E$ is representable is no easier than determining whether an ordinary preference admits a continuous utility representation—a well-studied problem in consumer theory. The appendix provides a necessary and sufficient condition for representability under E.1.

[3] Refer to Sen (1970) and d'Aspremont (1985) for more details.

Although E.1 applies to broad domains, E.3 is tailored to economic environments. It guarantees the existence of an egalitarian reference point at which all agents are equally well off. This requirement subsumes several familiar assumptions. For example, the assumption that $\left(\frac{\Omega}{n}, i\right) \sim_E \left(\frac{\Omega}{n}, j\right)$ for all i, j, expresses the idea that an equal division of resources is fair. In settings without preexisting ownership rights, equal division of common resources is likewise a natural benchmark, as in applications to carbon emission rights or collaborative resource use in polar or space exploration. E.3 also implies $(s, i) \sim_E (s, j)$ for all i, j, where $s \in \left(0, \frac{\Omega}{n}\right]$. When all agents receive the same bundle s, ensuring a basic standard of living, this serves as an analogue of List's (2001) "zero line of welfare."

Although E.3 does not explicitly require $(0, i) \sim_E (0, j)$ for all i, j, this condition implies E.3. By E.2, we can establish that $u_1(0) = \cdots = u_n(0)$. Assuming $u_1\left(\frac{\Omega}{n}\right) \leq \cdots \leq u_n\left(\frac{\Omega}{n}\right)$ without loss of generality, and noting that utility functions are continuous, we can identify z^* satisfying E.3. Under E.1, the assumption $(0, i) \sim_E (0, j)$ is equivalent to the requirement that $(x, i) \succcurlyeq_E (0, j)$ for all $i, j \in N$ and all $x \in R^l_+$, capturing the idea that, all else being equal, having no resources represents the most disadvantaged position.

The appendix demonstrates the independence of the three properties. Let $E(\succcurlyeq)$ denote the set of extended preferences satisfying E.1-3 generated from $\succcurlyeq$. Any representation of $\succcurlyeq$ by utility functions yields, via interpersonal utility comparisons, an extended preference. The following two examples illustrate extended preferences satisfying E.1–E.3, constructed using this approach.

Example 1 (Money-Metric Utility) Let $p \in int\Delta^l$ be an arbitrary price vector. Define the money-metric utility function as

$$u_i^p(x) = \min\{pq : q \sim_i x\}.$$

(See Weymark (1985) for further discussion.) Define the extended preference

$\succsim_E^p$ by

$$(x,i) \succsim_E^p (y,j) \Longleftrightarrow u_i^p(x) \geq u_j^p(y).$$

for any $x, y \in R_+^l$ and any $i, j \in N$. This extended preference satisfies property E.3. Let each agent be endowed with $\frac{\Omega}{t}$, where $t \geq n$ and let z_i^t be the optimal consumption bundle for agent i subject to the budget constraint $px \leq p\left(\frac{\Omega}{t}\right)$. Let $z^t = (z_i^t)_{i \in N}$. If t is sufficiently large, $z^t \in Z$, ensuring that E.3.holds.

Example 2 (Egalitarian-Equivalent Utility) Define the utility function as

$$u_i^\Omega(x) = \lambda_x, \text{ where } \lambda_x \text{ is such that } \lambda_x \Omega \sim_i x.$$

The extended preference $\succsim_E^\Omega$ is defined by:

$$(x,i) \succsim_E^\Omega (y,j) \Longleftrightarrow u_i^\Omega(x) \geq u_j^\Omega(y),$$

for any $x, y \in R_+^l$ and any $i, j \in N$.

The construction of Example 2 follows Fleurbaey and Maniquet (2011), although they do not use the terminology "extended preference", and it is closely related to egalitarian equivalence introduced by Pazner and Schmeidler (1978). A feasible allocation z is egalitarian-equivalent if and only if

$$(z_i, i) \sim_E^\Omega (z_j, j) \text{ for all } i, j.$$

When several extended preferences are used, we write $\succsim_{E'}$, $\succsim_{E''}$, and so on. Note the distinction between $\succsim_{E'}$ and $\succsim_E'$: the former denotes a different extended preference generated from $\succsim$, whereas the latter denotes the extended preference derived from a different underlying profile $\succsim'$. Further discussion appears in Sections 4 and 6.

2.3 The leximin criterion

Let $\succsim_E$ be an extended preference, which is fixed throughout this section. The *leximin criterion*, a lexicographic extension of the difference principle of Rawls

(1971), is defined as follows. Given an allocation z, order all pairs (z_i, i) in ascending order (from the worst-off to the best-off) according to $\succcurlyeq_E$, so that

$$(z_{i_1}, i_1) \preccurlyeq_E (z_{i_2}, i_2) \preccurlyeq_E \cdots \preccurlyeq_E (z_{i_{n-1}}, i_{n-1}) \preccurlyeq_E (z_{i_n}, i_n),$$

where ties, if any, are broken arbitrarily. For $k = 1, ..., n$, agent i_k is the k-th worst-off agent under allocation z; we denote this agent by $i_k(z)$. We now define a lexicographic order $\geq_{L(E)}$ on the set of allocations.

For any two allocations z and z',

(i) $z >_{L(E)} z'$ if and only if there exists $k \in \{1, ..., n\}$ such that

$$(z_{i_\tau(z)}, i_\tau(z)) \sim_E (z'_{i_\tau(z')}, i_\tau(z')) \text{ for all } \tau = 1, .., k-1$$

and

$$(z_{i_k(z)}, i_k(z)) \succ_E (z'_{i_k(z')}, i_k(z')).$$

(ii) $z =_{L(E)} z'$ if and only if

$$(z_{i_k(z)}, i_k(z)) \sim_E (z'_{i_k(z')}, i_k(z')) \text{ for all } k.$$

(iii) $z \geq_{L(E)} z'$ if and only if

$$z >_{L(E)} z' \text{ or } z =_{L(E)} z'.$$

Given two allocations z and z', z is *at least as leximin-just as* z' if $z \geq_{L(E)} z'$. If $z >_{L(E)} z'$, z is *more leximin-just than* z'; if $z =_{L(E)} z'$, z and z' are equally leximin-just. All these relations are well defined and transitive, and $\geq_{L(E)}$ is complete. A feasible allocation z is *leximin-equitable* if there is no other feasible allocation z' that is more leximin-just than z. Let $LME(\succcurlyeq_E)$ be the set of such allocations. By condition E.1, it follows that $LME(\succcurlyeq_E) \subset PO(\succcurlyeq)$. The subsequent section discusses the non-emptiness and structural properties of $LME(\succcurlyeq_E)$.

3 Properties of leximin-equitable allocations

Leximin-equitable allocations can be classified into two types. In the first type, every agent receives a positive bundle and all agents attain the same welfare level. We refer to this type as *egalitarian*. In the second type, which we call *singular*, some agents receive no goods, while their welfare is weakly higher than that of the remaining agents. The formal definitions are as follows. Let $z \in LME(\succcurlyeq_E)$ be given arbitrarily. Assume, without loss of generality, that $(z_1, 1) \preccurlyeq_E \cdots \preccurlyeq_E (z_n, n)$. An allocation z is said to be egalitarian if

$$(z_1, 1) \sim_E \cdots \sim_E (z_n, n), \text{ and } z_i \neq 0 \text{ for all } i.$$

An allocation z is said to be singular if there exists $k \geq 2$ such that

$$\overbrace{(z_1, 1) \sim_E \cdots \sim_E (z_{k-1}, k-1)}^{z_1, \ldots, z_{k-1} \neq 0} \preccurlyeq_E \overbrace{(z_k, k) \preccurlyeq_E \cdots \preccurlyeq_E (z_n, n)}^{z_k, \ldots, z_n = 0}.$$

The following theorem shows that, in the absence of condition E.3, both types may arise.

Theorem 1 *Assume that E.1 and E.2 hold. For any $z \in LME(\succcurlyeq_E)$, z is either egalitarian or singular.*

Proof. Take $z \in LME(\succcurlyeq_E)$ arbitrarily. Assume, without loss of generality, that

$$(z_1, 1) \preccurlyeq_E \cdots \preccurlyeq_E (z_n, n),$$

which can be rewritten as

$$u_1 (z_1) \leq \cdots \leq u_n (z_n),$$

using a representation $(u_i)_{i \in N}$ of $\succcurlyeq_E$. There are two cases to discuss.

Case 1. $z_i \neq 0$ for all i.

If $u_j(z_j) < u_{j+1}(z_{j+1})$ for some j, a sufficiently small transfer ε from $j+1$ to j leads to

$$u_1(z_1) \leq \cdots \leq u_j(z_j + \varepsilon) < u_{j+1}(z_{j+1} - \varepsilon) \leq \cdots \leq u_n(z_n).$$

This transfer improves z in the leximin sense, which yields a contradiction. Thus, $u_1(z_1) = \cdots = u_n(z_n)$, meaning that z is egalitarian.

Case 2. $z_i = 0$ for some i.

Note first that no agent j with $z_j \neq 0$ ranks (strictly) higher than i in this order. If so, the same proof as in Case 1 leads to a contradiction. Thus, we can assume

$$\overbrace{u_1(z_1) \leq \cdots \leq u_{k-1}(z_{k-1})}^{z_1,\ldots,z_{k-1} \neq 0} \leq \overbrace{u_k(z_k) \leq \cdots \leq u_n(z_n)}^{z_k,\ldots,z_n = 0}.$$

Here, we have $u_1(z_1) = \cdots = u_{k-1}(z_{k-1})$ as in Case 1, which implies that z is singular. ■

The singular type is difficult to justify from standard fairness considerations. Condition E.3 is both necessary and sufficient, in addition to E.1 and E.2, to eliminate singular types.

Theorem 2 *Assume E.1 and E.2. E.3 is necessary and sufficient for leximin-equitable allocations to exist and for all of them to be egalitarian.*

Proof. Since the necessity part is straightforward, we prove sufficiency.

Existence: Let $(u_i)_{i \in N}$ be an arbitrary representation of $\succcurlyeq_E$. Define the set E of utility vectors by

$$(u_1, \ldots, u_n) \in E \Longleftrightarrow u_1 = \cdots = u_n \text{ and } \exists z \in Z \text{ s.t. } u_i = u_i(z_i)\ \forall i.$$

By E.3, E is well-defined. Consider the 45-degree line in R^n passing through the origin. All points of E are on this line. The maximal point $\widehat{u}$ of E exists on this line because E is closed and upper-bounded. We show that $\widehat{u}$ corresponds to an egalitarian allocation, which completes the proof of existence.

Let $\widehat{z}$ be the feasible allocation attaining $\widehat{u}$. Letting z_i^* satisfy E.3, we have $u_i(\widehat{z}_i) \geq u_i(z_i^*)$. As $z_i^* \neq 0$, the monotonicity of preference implies $u_i(\widehat{z}_i) > u_i(0)$. Now, we show $\widehat{z} \in PO(\succcurlyeq)$. Suppose not. Then, there exists some $z' \in Z$ such that $u_i(z_i') > u_i(\widehat{z}_i)$ for all i. Without loss of generality, we assume that agent n has the lowest utility in z'. Thus, for any $i \neq n$, $u_i(z_i') \geq u_n(z_n') > u_i(\widehat{z}_i)$. By removing a small amount ε_i of goods from z_i', we can make $u_i(z_i' - \varepsilon_i) = u_n(z_n')$, which is possible by the intermediate value theorem on $[z_i', 0]$, noting $u_i(\widehat{z}_i) > u_i(0)$ as shown above. Let $z^{-\varepsilon_i}$ be the feasible allocation created this way. By definition, the utility vector attained by $z^{-\varepsilon_i}$ lies on E, and to the upper right of $\widehat{u}$, which is a contradiction. Thus, we obtain $\widehat{z} \in PO(\succcurlyeq)$.

Next, we show that $\widehat{z}$ is leximin-equitable. Suppose not. Then, there exists some $z \in Z$ that is more leximin just than $\widehat{z}$. The diagram below illustrates the comparison of utilities in $\widehat{z}$ and z.

$$\begin{array}{ccccccccccccc} u_1(\widehat{z}_1) & = & \cdots & = & u_{k-1}(\widehat{z}_{k-1}) & = & u_k(\widehat{z}_k) & = & \cdots & = & u_n(\widehat{z}_n) \\ \parallel & & & & \parallel & & \wedge & & & & \\ u_{i_1}(z_{i_1}) & \leq & \cdots & \leq & u_{i_{k-1}}(z_{i_{k-1}}) & \leq & u_{i_k}(z_{i_k}) & \leq & \cdots & \leq & u_{i_n}(z_{i_n}) \end{array}$$

It follows that $u_i(z_i) \geq u_i(\widehat{z}_i)$ for all i, with strict inequality for $i_k, ..., i_n$, which contradicts $\widehat{z} \in PO(\succcurlyeq)$.

All are egalitarian: To ensure that all of them are egalitarian, take $z \in LME(\succcurlyeq_E)$ arbitrarily. Suppose that $(z_1, 1) \preccurlyeq_E \cdots \preccurlyeq_E (z_n, n)$ without loss of generality. On the other hand, E.3 means $(z_1^*, 1) \sim_E \cdots \sim_E (z_n^*, n)$. As z is at least as leximin just as z^*, we have $(z_1^*, 1) \preccurlyeq_E (z_1, 1)$, which implies $(z_i^*, i) \preccurlyeq_E (z_i, i)$ for all i. Thus, E.1 implies $z_i^* \preccurlyeq_i z_i$ for all i. As $z_i^* \neq 0$, z_i is so as well, which holds for all i. Therefore, by Theorem 1, z is egalitarian. ■

Henceforth, we assume E.1-3.

The leximin criterion satisfies Hammond equity and its variants.[4] Hammond equity judges that allocations exhibiting smaller inter-individual utility disparities be regarded as more desirable. Such disparities can be eliminated owing to the complete divisibility of goods and the continuity of preferences. Consequently, all leximin-equitable allocations are egalitarian.

4 Price-indicator type of extended preferences

Suppose $z \in PO(\succsim, p)$. An extended preference $\succsim_E$ is said to be a *price-indicator type* with respect to z and p if for any $i, j \in N$ and any $x \in R^l_+$,

$$(x, j) \succsim_E (z_i, i) \implies px \geq pz_i, \text{ and } (x, j) \succ_E (z_i, i) \implies px > pz_i.$$

Note that $(z_i, i) \succsim_E (z_j, j)$ holds if and only if $pz_i \geq pz_j$. Intuitively, the price-indicator-type extended preference ranks agents' welfare at z according to their wealth levels, as measured by pz_i; an agent is considered better off if she is wealthier.

Remark 1 *For any $z \in PO(\succsim, p)$, the extended preference described in Example 1 is a price-indicator type with respect to z and p. In contrast, the extended preference introduced in Example 2 is not, as will be shown later.*

Theorem 3 below establishes a relationship between leximin-equitable allocations and EIW allocations through the notion of a price-indicator type.

Theorem 3 *Let $z \in PO(\succsim, p)$ and $\succsim_E \in E(\succsim)$ be given. Any two of the following three conditions imply the third.*

(i) $z \in EIW(\succsim, p)$.

[4] Hammond equity was first introduced by Hammond (1976). A slightly stronger version (XE) was proposed by Deschamps and Gevers (1978) and d'Aspremont and Gevers (1976). The weakest version, minimal equity (ME), was also introduced by d'Aspremont and Gevers (1977).

(ii) $z \in LME(\succcurlyeq_E)$.

(iii) $\succcurlyeq_E$ *is a price-indicator type at* p *and* z.

Proof. (i) and (ii) imply (iii).

Suppose $(x, j) \succcurlyeq_E (z_i, i)$. By Theorem 2 and condition (ii), we have $(z_i, i) \sim_E (z_j, j)$ for all i, j. Hence, $(x, j) \succcurlyeq_E (z_j, j)$, which implies $x \succcurlyeq_j z_j$ by condition E.1. Therefore, by condition (i), we have $px \geq pz_j$. An analogous argument shows that $(x, j) \succ_E (z_i, i)$ implies $px > pz_j$, completing the proof.

(ii) and (iii) imply (i).

By Theorem 2 and condition (ii), we have $(z_i, i) \sim_E (z_j, j)$ for all i, j. Since $\succcurlyeq_E$ is the price-indicator type with respect to p and z, this equivalence implies $pz_i = pz_j$ for all i, j. Together with $z \in PO(\succcurlyeq, p)$, this yields $z \in EIW(\succcurlyeq)$.

(iii) and (i) imply (ii).

Suppose, to the contrary, that $(z_j, j) \succ_E (z_i, i)$ for some i, j. By condition (iii), this implies $pz_j > pz_i$, contradicting $pz_i = pz_j$, which follows directly from condition (i). Therefore, $(z_i, i) \sim_E (z_j, j)$ for all i, j. Since $z \in PO(\succcurlyeq)$, Theorem 2 implies that $z \in LME(\succcurlyeq_E)$. ■

It is important not to misinterpret Theorem 3 as establishing equivalence among conditions (i)–(iii). Rather, it asserts that any two of the three imply the remaining one, namely,

$$\text{(i)+(ii)} \Longrightarrow \text{(iii)}, \ \text{(iii)+(i)} \Longrightarrow \text{(ii)}, \text{ and } \text{(ii)+(iii)} \Longrightarrow \text{(i)}.$$

Condition (i) alone does not imply either (ii) or (iii), since the class $E(\succcurlyeq)$ is rich. The same observation applies symmetrically to the other conditions. Further discussion can be found in the Appendix. Theorem 3 admits the following interpretation.

First, conditions (i) and (ii) imply (iii): if an extended preference renders an EIW allocation leximin-equitable, then it must be a price-indicator type. Second, conditions (ii) and (iii) imply (i): if a feasible allocation is leximin-equitable with respect to an extended preference that is a price-indicator type, then it is an EIW allocation. Finally, conditions (iii) and (i) imply (ii): given an EIW allocation, if one constructs an extended preference that is a price-indicator type using the associated equilibrium prices, then that allocation becomes leximin-equitable under the constructed preference.

Prior to our study, Fleurbaey and Maniquet (2008, 2011) investigated the relationship between leximin-equitable and EIW allocations. They also consider the leximin criterion, which is defined using $\succsim_E^\Omega$ introduced in Example 2 and termed Ω-Equivalent Leximin ($R^{\Omega lex}$, p. 7). Note that $\succsim_E^\Omega$ is not, in general, a price-indicator type. Take an arbitrary $z \in EIW(\succsim, p)$. If $\succsim_E^\Omega$ were a price-indicator type with respect to p and z, Theorem 3 would imply that z is a leximin-equitable allocation. Together with Theorem 2 and the definition of $\succsim_E^\Omega$, this would further imply that all agents' indifference curves passing through their consumption at z intersect at the same point on the line between the origin and Ω, which is an implausible implication. The same reasoning applies if one instead assumes that z is leximin-equitable.

Hence, the leximin-equitable allocations defined using $\succsim_E^\Omega$ do not necessarily coincide with the EIW allocations. This conclusion is consistent with Table 5.1 in Fleurbaey and Maniquet (2011, p. 95), which summarizes whether the EIW rule and the Leximin rule satisfy various axioms. As shown in the table, the two rules display different patterns of signs.

Moreover, Fleurbaey and Maniquet (2008) introduce the Ω-Implicit Income Maximin relation, which is defined through a sophisticated construction based on the money-metric utility function appearing in Example 1. Under a variable population setting, they show that this relation renders the set of EIW allocations rationalizable. By contrast, the leximin relation $\geq_{L(E)}$ cannot necessarily do so: If the leximin relation rendered the set of EIW allocations rationalizable, Theorem 2 would imply that all the EIW allocations share the same equilibrium prices, which is highly restrictive.

5 Informational Basis for Rules

In the literature on SWFLs, a rule f is defined as follows:

$$\succsim_E \quad \longrightarrow \quad f(\succsim_E), \tag{1}$$

where $f(\succsim_E)$ is a non-empty subset of feasible allocations. Under this definition, the EIW rule and the Leximin rule are distinct: the former associates each extended preference $\succsim_E$ with the set of EIW allocations $EIW(\succsim)$, whereas the latter associates it with the set of leximin-equitable allocations $LME(\succsim_E)$.[5] Theorem 3 indicates that as long as $\succsim_E$ is not of the price-indicator type, the equivalence of the two sets of allocations is not guaranteed.

To provide an axiomatization of the EIW rule grounded in leximin fairness, we adopt the following strategy. A rule F is defined in the standard way: a mapping from each profile $\succsim \in Q^n$ to a non-empty subset $F(\succsim)$ of Z. The outcome $F(\succsim)$ is determined by comparing the agents' welfare levels. We take agents' preferences as the only primitive objects, and regard extended preferences as auxiliary constructs derived from them. Nonetheless, they serve as indispensable tools for determining $F(\succsim)$. The figure below illustrates this relationship.

$$\succsim \qquad \overset{\succsim_E, \succsim_{E'}, \ldots \in E(\succsim)}{\overset{\downarrow}{\longrightarrow}} \qquad F(\succsim) \tag{2}$$

All EIW allocations can be regarded as equally fair, since they result from free choices by agents who start from identical endowments and face the same

[5] The axiomatic characterization of the Leximin rule within the SWFL framework was established by d'Aspremont and Gevers (1977), Deschamps and Gevers (1978), Hammond (1976), and Sen (1977). Yamamura (2017) subsequently generalized these results, thereby revealing a more fundamental structure underlying the axiomatization.

prices. Accordingly, as shown in Theorem 3, EIW allocations are leximin-equitable only when they are evaluated using price-indicator-type extended preferences.

Theorem 3 further implies that a single extended preference can render all EIW allocations leximin-equitable if and only if equilibrium prices are unique—an assumption that is generally implausible. Hence, one must rely on multiple extended preferences, which may give rise to mutually conflicting leximin judgments. These observations motivate the following formulation.

Let $D(\succcurlyeq, z)$ be a non-empty subset of $E(\succcurlyeq)$. If z is Pareto optimal, then $D(\succcurlyeq, z)$ consists of the price-indicator-type extended preferences with respect to z and p. Here, p is a supporting price vector. Note that it does not necessarily include every price-indicator type, and p may not be unique. Moreover, if z is an EIW allocation, p must also be an equilibrium price vector. We interpret $D(\succcurlyeq, z)$ as the *informational basis* for assessing whether $z \in F(\succcurlyeq)$. Furthermore, $D(\succcurlyeq, z)$ is well defined, as demonstrated in Example 1.

In general, each $D(\succcurlyeq, z)$ comprises multiple extended preferences, each of which may generate distinct leximin judgments, as discussed above. A mechanism is therefore required to aggregate these judgments. The following two conditions are minimal requirements for such a mechanism:

1. For z to be selected, it must be leximin-equitable under at least one extended preference. Otherwise, z would be chosen independently of the leximin criterion.

2. If z is leximin-equitable under all extended preferences, it must be selected; otherwise, the leximin criterion would have no discriminative power.

Leximin Justification Possibility (LMJP) formalizes the first principle, and Leximin All Unanimity (LMAU) formalizes the second. A rule F satisfies LMJP if, for any $z \in F(\succcurlyeq)$, there exists some $\succcurlyeq_E \in D(\succcurlyeq, z)$ such that $z \in LME(\succcurlyeq_E)$. A rule F satisfies LMAU if $z \in F(\succcurlyeq)$ whenever $z \in LME(\succcurlyeq_E)$ holds for all

$\succsim_E \in D(\succsim, z)$. These two conditions provide the most natural ways to aggregate leximin judgments in situations where a single judgment is insufficient and multiple judgments may conflict. Let

$$LME^{\exists}(\succsim) = \left\{ z \in Z : z \in \bigcup_{\succsim_E \in D(\succsim, z)} LME(\succsim_E) \right\}$$
$$LME^{\forall}(\succsim) = \left\{ z \in Z : z \in \bigcap_{\succsim_E \in D(\succsim, z)} LME(\succsim_E) \right\}.$$

Here, $LME^{\forall}(\succsim_E)$ is non-empty. Take $z \in EIW(\succsim)$. By the definition of $D(\succsim, z)$, each $\succsim_E \in D(\succsim, z)$ is of the price-indicator type with respect to z and p, where p is an equilibrium price vector of z. Thus, by Theorem 3, we have $z \in LME(\succsim_E)$. As this holds for all $\succsim_E \in D(\succsim, z)$, it follows that $z \in LME^{\forall}(\succsim)$, as required. Hence, $LME^{\exists}(\succsim)$ is also non-empty.

Using these notations, we can provide more concise definitions of LMJP and LMAU.

$$\begin{array}{rcl} \text{LMJP} & : & F(\succsim) \subset LME^{\exists}(\succsim) \text{ for all } \succsim \in Q^n. \\ \text{LMAU} & : & LME^{\forall}(\succsim) \subset F(\succsim) \text{ for all } \succsim \in Q^n. \end{array}$$

Theorem 4 *$LME^{\forall}(\succsim) = EIW(\succsim) = LME^{\exists}(\succsim)$ for all $\succsim \in Q^n$.*

Proof. The following three inclusions establish the result:

(i) $LME^{\forall}(\succsim) \subset EIW(\succsim)$,

(ii) $EIW(\succsim) \subset LME^{\exists}(\succsim)$, and

(iii) $LME^{\exists}(\succsim) \subset LME^{\forall}(\succsim)$.

Among these, (i) and (ii) follow directly from the definition of $D(\succsim, z)$ and from Theorem 3; thus, it suffices to prove (iii). Take $z \in LME^{\exists}(\succsim)$. Then, $z \in LME(\succsim_E)$, where $\succsim_E$ is a price-indicator type with respect to z and its supporting price vector p. By Theorem 3, $z \in EIW(\succsim)$. From the definition of $D(\succsim, z)$, every $\succsim_{E'} \in D(\succsim, z)$ is a price-indicator type with respect to z and some equilibrium price p' associated with z. Applying Theorem 3 once more, we obtain $z \in LME(\succsim_{E'})$, which completes the proof. ■

The following theorem provides the leximin axiomatization of the EIW rule. The proof is omitted, as it follows directly from Theorem 4.

Theorem 5 *The EIW rule is the only rule satisfying LMJP and LMAU.*

The axioms are independent. Any strict subcorrespondence of the EIW rule satisfies LMJP but not LMAU. In contrast, the all-select rule—which selects all feasible allocations for every profile—satisfies LMAU but not LMJP.

6 Locally created extended preferences

This section provides an alternative characterization of the price-indicator-type extended preference. We consider extended preferences constructed solely from local information. The notion of local creation introduced here is analogous to local independence (Nagahisa 1991; Nagahisa and Suh 1995). Let $p_N = (p_i)_{i \in N}$, where $p_i \in int\Delta^l$, and let $z \in Z$. We say that two preference profiles $\succsim$ and $\succsim'$ are p_N-identical at z if, for every i,

$$x \succsim_i z_i \Longrightarrow p_i x_i \geq p_i z_i, \text{ and } x \succsim'_i z_i \Longrightarrow p_i x_i \geq p_i z_i,$$

that is, p_i serves as a common supporting price vector for both $\succsim_i$ and $\succsim'_i$ at z_i. Let $p_N(\succsim, z)$ denote the set of profiles that are p_N-identical to $\succsim$ at z. If $\succsim$ and $\succsim'$ are p_N-identical at z, then $p_N(\succsim, z) = p_N(\succsim', z)$; hence, we write $p_N(\cdot, z)$ whenever the reference preference is immaterial.

For an extended preference $\succsim_E$, the welfare ranking of $\succsim_E$ at z, denoted by $\geq_{(\succsim_E, z)}$, is the order on N defined by

$$i \geq_{(\succsim_E, z)} j \Longleftrightarrow (z_i, i) \succsim_E (z_j, j),$$

for all i, j. This ranking identifies agents' relative welfare positions at z.

Now, define

$$\bigcap_{\succcurlyeq \in p_N(\cdot,z)} \left\{ \geq_{(\succcurlyeq_E, z)} : \succcurlyeq_E \in E(\succcurlyeq) \right\} \tag{3}$$

or equivalently,

$$\bigcap_{\succcurlyeq' \in p_N(\succcurlyeq,z)} \left\{ \geq_{(\succcurlyeq'_E, z)} : \succcurlyeq'_E \in E(\succcurlyeq') \right\}.$$

We say that an extended preference $\succcurlyeq_E$ is locally created at z and p_N if and only if $\geq_{(\succcurlyeq_E,z)}$ belongs to the set in (3). Then, $\geq_{(\succcurlyeq_E,z)}$, the welfare ranking at z, is determined only by preferences in a neighborhood of z and does not depend on global information far from z.

The intuition is particularly transparent when individual preferences are smooth and z lies in the interior. Then p_N matches the marginal rates of substitution (MRS) at z, which are equal across all $\succcurlyeq' \in p_N(\succcurlyeq, z)$. Hence, these preferences are locally identical in a neighborhood of z. If the MRSs at z are the only available information for constructing extended preferences, they provide the same basis for determining the welfare ranking at z. Thus, the set of welfare rankings at z is identical across all $\succcurlyeq' \in p_N(\succcurlyeq, z)$, and consequently, the welfare rankings derived from a locally created extended preference belong to (3).

We say that an extended preference is locally created at z if, for some p_N, it becomes a locally created one at z and p_N. If z is irrelevant as well, we simply omit "at z".

Without assuming smoothness or interiority, different supporting prices $p'_N = (p'_i)_{i \in N}$ may define different sets as in (3). However, when preferences are smooth and interiority is assumed, this multiplicity disappears.

Locally created extended preferences are analogous to local independence (LI) (Nagahisa 1991; Nagahisa and Suh 1995), which was used to characterize the Walras rule. LI is a consistency condition in social choice: it requires that the choice of z remain the same under $\succcurlyeq$ and $\succcurlyeq'$ whenever the equality of marginal rates of substitution (MRS) holds at z. In contrast, the notion of locally created extended preferences imposes a much weaker requirement. It only requires that the welfare ranking at z remain unchanged, without implying that the choice of z itself must coincide under $\succcurlyeq$ and $\succcurlyeq'$.

From now on, we impose the following additional assumption on $E(\succcurlyeq)$. If every agent has the same preference, we write $\succcurlyeq = (\succcurlyeq, ..., \succcurlyeq)$, by a slight abuse of notation. Let $\succcurlyeq_{E^*}$ denote the extended preference such that, for any $i, j \in N$ and any $x, y \in R^l_+$,

$$(x, i) \succcurlyeq_{E^*} (y, j) \Leftrightarrow x \succcurlyeq y.$$

We then modify $E(\succcurlyeq)$ as follows.

D.1. If every agent has the same preference, then $E(\succcurlyeq) = \{\succcurlyeq_{E^*}\}$.[6]

There is no compelling reason to reject D.1. If all agents share the same preference, that preference should be the only possible extended preference, and no other extension should be considered. Apart from this special case, $E(\succcurlyeq)$ remains unchanged.

Lemma 1 *Suppose $z \in PO(\succcurlyeq, p)$, and, without loss of generality, $pz_1 \leq pz_2 \leq \cdots \leq pz_n$. Let $\geq^p$ denote the welfare ranking at z induced by this ordering. Then:*

(i) Any extended preference that is locally created at z and p yields the same welfare ranking as $\geq^p$. Formally,

$$\{\geq^p\} = \overbrace{\bigcap_{\succcurlyeq' \in p(\succcurlyeq, z)} \Big\{\geq_{(\succcurlyeq'_E, z)} : \succcurlyeq'_E \in E(\succcurlyeq')\Big\}}^{\text{the set of (3)}}. \tag{4}$$

[6] Note that $\succcurlyeq_{E^*}$ satisfies E.1–E.3. D.1 and E.1–E.3 belong to different categories: E.1–E.3 are properties of extended preferences, whereas D.1 is a property of the *set* of extended preferences.

(ii) If $\succsim_E \in E(\succsim)$ is of the price-indicator type with respect to z and p, then it is locally created at z and p.

(iii) If $z \in EIW(\succsim, p)$, then the converse of (ii) also holds; that is, any extended preference that is locally created at z and p is of the price-indicator type with respect to z and p.

Proof. Let $\succsim^p$ be a profile in which each agent's preference is represented by the utility function $u(x) = px$.

Consider any $\succsim' \in p(\succsim, z)$. Since the extended preference $\succsim'_E$ defined by money-metric utility (Example 1) induces the welfare ranking $\geq^p$, it follows that

$$\geq^p \in \left\{ \geq_{(\succsim'_E, z)} : \succsim'_E \in E(\succsim') \right\}.$$

As this inclusion holds for every $\succsim' \in p(\succsim, z)$, we obtain

$$\geq^p \in \overbrace{\bigcap_{\succsim' \in p(\succsim, z)} \left\{ \geq_{(\succsim'_E, z)} : \succsim'_E \in E(\succsim') \right\}}^{\text{the set of (3)}}.$$

Furthermore, by condition D.1 we have

$$\{\geq^p\} = \left\{ \geq_{(\succsim^p_E, z)} : \succsim^p_E \in E(\succsim^p) \right\}.$$

Since $\succsim^p \in p(\succsim, z)$, combining these two relations yields (4) and completes the proof of (i). Statements (ii)-(iii) are established on the basis of (4).

(ii) Suppose that $\succsim_E \in E(\succsim)$ is of the price-indicator type with respect to z and p. It is straightforward to verify that the associated welfare ranking $\geq_{(\succsim_E, z)}$ coincides with $\geq^p$. Hence, by (4), the desired claim holds.

(iii) Suppose that $\succsim_E \in E(\succsim)$ is locally created at z and p. Since $z \in EIW(\succsim, p)$, it follows that $pz_1 = pz_2 = \cdots = pz_n$. By (i), we therefore obtain $(z_1, 1) \sim_E (z_2, 2) \sim_E \cdots \sim_E (z_n, n)$. Taken together, this implies that $(x, j) \succsim_E (z_i, i) \Longrightarrow px_j \geq pz_i$ as shown below.

$$(x, j) \succsim_E (z_i, i) \overset{(z_i,i) \sim_E (z_j,j)}{\overset{\downarrow}{\Longrightarrow}} (x, j) \succsim_E (z_j, j) \overset{E.1}{\overset{\downarrow}{\Longrightarrow}} x \succsim_j z_j \overset{z \in EIW(\succsim, p)}{\overset{\downarrow}{\Longrightarrow}} px \geq pz_j \overset{pz_j = pz_i}{\overset{\downarrow}{\Longrightarrow}} px_j \geq pz_i$$

Similarly, we have $(x, j) \succ_E (z_i, i) \Longrightarrow px_j \geq pz_i$. These observations complete the proof of (iii). ■

Note that (ii) ensures the existence of locally created extended preferences, since price-indicator-type extended preferences exist. Under assumption D.1, the informational basis is redefined as a set of extended preferences that are locally created at z.

7 Discussion

This section addresses three remaining issues.

(1) Smooth economies

All results extend to smooth economies with a minor modification. Let $P \subset Q$ consist of smooth preferences that satisfy the boundary condition

$$\left\{0 \leq x \leq \Omega : x \succsim \frac{\Omega}{n}\right\} \subset R^l_{++},$$

which ensures that EIW allocations are interior. Strengthen E.3 as follows:

E.3' There exists $z^* \in Z$ such that

$$(z_1^*, 1) \sim_E (z_2^*, 2) \sim_E \cdots \sim_E (z_n^*, n),$$

and for all i,

$$\{0 \leq x \leq \Omega : x \succsim_i z_i^*\} \subset R^l_{++}.$$

Under E.3', every leximin equitable allocation is interior. Indeed, if $z \in LME(\succsim_E)$, Theorem 2 gives

$$(z_1, 1) \sim_E (z_2, 2) \sim_E \cdots \sim_E (z_n, n),$$

and leximin comparison with z^* yields $z_i \succsim_i z_i^*$ for all i, implying $z \in R^{nl}_{++}$.

Taking $E(\succcurlyeq)$ as the set of extended preferences satisfying E.1, E.2, and E.3' preserves D.1, and Remark 1 ensures their existence. The only adjustment in the proofs is at the start of Lemma 1, where $\succcurlyeq^p$ is replaced by a CES-based profile satisfying the boundary condition.

(2) The relationship with the axiomatization of Nagahisa and Suh (1995)

They establish that, under smooth preferences and an interiority condition, the EIW rule is uniquely characterized by Pareto Optimality (PO), Individual Rationality (IR), and Local Independence (LI). The axiom LI is formulated as follows.

Consider two profiles, $\succcurlyeq^1$ and $\succcurlyeq^2$, and a feasible allocation $z \in R^{nl}_{++}$. Suppose that, for every agent i, the marginal rates of substitution (MRS) associated with $\succcurlyeq_i$ and $\succcurlyeq'_i$ coincide at z_i. Under this condition, LI requires that

$$z \in F(\succcurlyeq^1) \Longleftrightarrow z \in F(\succcurlyeq^2). \tag{5}$$

The leximin-based axioms LMJP and LMAU jointly implement LI when the profiles are evaluated on the informational basis. They achieve (5) through the following chain of equivalences

$$z \in F(\succcurlyeq^1) \Longleftrightarrow z \in EIW(\succcurlyeq^1) \Longleftrightarrow z \in EIW(\succcurlyeq^2) \Longleftrightarrow z \in F(\succcurlyeq^2). \tag{6}$$

The middle implication in (6) is immediate because the MRS at z are identical under the two profiles. The left- and right-hand equivalences are established in the proof of Theorem 5.

Next, consider equation (6) under the MRS-equality condition. Since z is Pareto optimal with respect to both $\succcurlyeq^1$ and $\succcurlyeq^2$, the two profiles admit a common supporting price vector p that uniquely supports z. Without loss of generality, impose the ordering $pz_1 \geq \cdots \geq pz_n$, and let $\geq^p$ denote the welfare ranking at z induced by this price-based ordering of the agents.

Lemma 1(i) then yields

$$\overbrace{\bigcap_{\succsim' \in p(\succsim^1, z)} \left\{ \geq_{(\succsim'_E, z)} : \succsim'_E \in E(\succsim') \right\}}^{\text{the set of (3)}} = \{\geq^p\} = \overbrace{\bigcap_{\succsim' \in p(\succsim^2, z)} \left\{ \geq_{(\succsim'_E, z)} : \succsim'_E \in E(\succsim') \right\}}^{\text{the set of (3)}}. \tag{7}$$

Furthermore, Lemma 1(ii) implies that both $D\left(\succsim^1, z\right)$ and $D\left(\succsim^2, z\right)$ consist exclusively of extended preferences that are locally created at z and p. Combining this observation with (7), it follows that every $\succsim^1_E \in D\left(\succsim^1, z\right)$ and every $\succsim^2_E \in D\left(\succsim^2, z\right)$ induces $\geq^p$ as its welfare ranking at z. By Theorem 3, LMJP and LMAU therefore render identical judgment under both $\succsim^1$ and $\succsim^2$. In other words, when the two profiles agree on the MRS at z, the leximin-based evaluation remains invariant.

(3) Relevance to Invariance axioms

Consider two profiles $\succsim$ and $\succsim'$ that are ordinally identical; that is, for all $i \in N$ and all $x, y \in R^l_+$,

$$x \succsim_i y \text{ if and only if } x \succsim'_i y.$$

Under such profiles, the set of EIW allocations is invariant across all corresponding extended preferences $\succsim_E \in E(\succsim)$ and $\succsim'_E \in E(\succsim')$. In contrast, leximin-equitable allocations need not satisfy this invariance property: it may happen that $z \in LME(\succsim_E)$ while $z \notin LME(\succsim'_E)$. This distinction is closely related to the role of invariance axioms in the theory of social welfare functionals (SWFLs).[7] In particular, the EIW rule satisfies the Ordinal and Non-Comparability (ONC) axiom—the strongest among the standard invariance requirements—whereas the leximin criterion fails to do so.

This observation indicates that LMJP and LMAU may impede an axiomatization of the EIW rule: it can arise that an allocation z qualifies as an EIW

[7] Virtually all of the literature on SWFLs addresses invariance axioms. These axioms admit several versions, ranging from the weakest to the strongest. See, for example, d'Aspremont and Gevers (1977), which provides a useful illustration of their logical relationships.

allocation, yet no leximin judgment identifies z as leximin-equitable, while every leximin judgment selects some non-EIW allocation instead. The axiomatization established in Theorem 6 circumvents this difficulty by restricting the admissible domain of extended preferences. Specifically, LMJP and LMAU are applied only to price-indicator-type extended preferences.

Let $\succsim_E$ and $\succsim_{E'}$ be of the price-indicator types with respect to z and p. The figure below illustrates that

$$z \in LME(\succsim_E) \overset{\substack{\text{Th.3} \\ \downarrow}}{\Longrightarrow} z \in EIW(\succsim, p) \Longrightarrow z \in EIW(\succsim', p) \overset{\substack{\text{Th.3} \\ \downarrow}}{\Longrightarrow} z \in LME(\succsim'_E).$$

The central implication follows directly from the fact that $\succsim'_E$ belongs to the class of price-indicator types with respect to z and p.

In view of Lemma 1, restricting the class of extended preferences to price-indicator types may be interpreted, at the semantic level, as a relaxation of Arrow's Independence of Irrelevant Alternatives (IIA) within the framework of SWFLs. Baccelli (2023) demonstrates that such a relaxation opens an alternative path toward interpersonal welfare comparisons within the Arrovian paradigm, distinct from the approaches grounded in invariance axioms.[8]

8 Conclusion

This paper connects two strands of research that have developed largely independently within social choice theory: interpersonal comparisons of welfare and the axiomatic analysis of resource-allocation problems. The no-envy approach has played a central role in the axiomatization of the EIW rule.[9] Related contributions include Fleurbaey and Maniquet (2011), who study the normative foundations of the EIW rule using social ordering functions, and Chambers and

[8] Baccelli (2023) provides an extensive analysis of the Borda and relative utilitarian rules along this alternative route. See also Fleurbaey (2003) for a comprehensive account of informational bases through the notion of data filters.

[9] Thomson (2011) surveys justifications based on the no-envy concept.

Hayashi (2017), who provide an alternative axiomatization of the Walras rule for income-distribution problems. This paper contributes to this literature by clarifying the relationship between leximin equity and the EIW rule.

While it is well known that the EIW rule satisfies no envy, the case of the leximin solution is more subtle. This paper establishes a previously unnoticed and non-trivial link between leximin and EIW allocations.

9 Appendix

Independence of Axioms E.1-E.3

We establish the logical independence of axioms E.1–E.3 by constructing, for each axiom, an extended preference that satisfies the remaining two axioms while violating the axiom under consideration.

Independence of Axiom E.1.

Consider a profile $\succcurlyeq$ in which individual preferences are not representable by linear utility functions. Fix a price vector $p \in \Delta^l$, and define an extended preference $\succcurlyeq_E$ by

$$(x, i) \succcurlyeq_E (y, j) \Longleftrightarrow px \geq py.$$

Then $\succcurlyeq_E$ satisfies E.2 and E.3, since it admits a (linear) representation and allows for an egalitarian reference bundle. However, it violates E.1. Indeed, if E.1 were satisfied, then for each agent i, the restriction of $\succcurlyeq_E$ to pairs (x, i) and (y, i) would coincide with the underlying preference $\succcurlyeq_i$. This would imply that $\succcurlyeq_i$ admits the linear utility representation $u(x) = px$, contradicting the assumption that individual preferences are not representable by linear utility functions. Hence, Axiom E.1 is independent of E.2 and E.3.

Independence of Axiom E.2.

Let utility functions be given by

$$v_i(x) = \sum_{h=1}^{l} x_{ih} + 1, \ i = 1, ..., n$$

and let $\succcurlyeq$ denote the profile induced by these functions.

Now modify agent 1's utility function by setting her utility equal to 0 at $x = 0$, while leaving it unchanged otherwise. Denote this modified utility function by v_1'. Let $\succcurlyeq_E$ be the extended preference constructed via interpersonal utility comparisons using v_1' for agent 1 and v_i for agents $i = 2, ..., n$. By construction, $\succcurlyeq_E$ satisfies E.1, since each agent's extended preference coincides with her underlying preference, and it satisfies E.3, as an egalitarian reference bundle exists. We claim that $\succcurlyeq_E$ violates E.2. Suppose, for contradiction, that $\succcurlyeq_E$ admits a representation $(u_i)_{i \in N}$. For all $x \neq 0$, we have $(x, 1) \sim_E (x, 2)$, which implies $u_1(x) = u_2(x)$. Since both u_1 and u_2 are continuous, taking the limit as $x \longrightarrow 0$ yields $u_1(0) = u_2(0)$, and hence $(0, 1) \sim_E (0, 2)$. This contradicts the definition of $\succcurlyeq_E$, under which $(0, 1) \prec_E (0, 2)$. Therefore, $\succcurlyeq_E$ does not admit a representation, and E.2 is violated. Hence, Axiom E.2 is independent of E.1 and E.3.

Independence of Axiom E.3.

Let u_i be agent i's continuous utility function representing $\succcurlyeq_i$, and suppose $u_1(0) > u_2(\Omega)$. Let $\succcurlyeq_E$ be the extended preference constructed via interpersonal utility comparisons using the utilities u_i. By construction, $\succcurlyeq_E$ satisfies E.1 and E.2. However, it violates E.3, since no feasible allocation can equalize welfare across agents: even if agent 2 receives the entire endowment, her utility remains strictly below agent 1's utility at the zero bundle. Hence, Axiom E.3 is independent of E.1 and E.2.

Independence of Conditions (i)–(iii) in Theorem 3

We establish the logical independence of conditions (i)–(iii) in Theorem 3.

(i) does not imply (ii).

Let $z \in EIW(\succcurlyeq, p)$, and let $(u_i)_{i \in N}$ be a representation of $\succcurlyeq_E$ such that not all the values $u_i(z_i)$ are equal. By Theorem 2, it follows that $z \notin LME(\succcurlyeq_E)$.

(i) does not imply (iii).

Using the same example, suppose that $\succcurlyeq_E$ is a price-indicator type with respect to z and p. Then by Theorem 3, we must have $z_i \sim_E z_j$ for all i, j, which contradicts the assumption that not all the values $u_i(z_i)$ are equal.

(ii) does not imply (i).

Let z be such that $z \in PO(\succcurlyeq)$, $z_i \neq 0$ for all i, and $z \notin EIW(\succcurlyeq)$. Define an extended preference $\succcurlyeq_E$ whose representation satisfies $u_i(z_i) = 1$ for all i. Then, by Theorem 2, we have $z \in LME(\succcurlyeq_E)$ while $z \notin EIW(\succcurlyeq)$.

(ii) does not imply (iii).

Let z and p be such that $z \in PO(\succcurlyeq, p)$, $z_i \neq 0$ for all i, and not all the values pz_i are equal. Using money-metric utility functions u_i^p, define an extended preference by

$$(x, i) \succcurlyeq_E (y, j) \Longleftrightarrow \alpha_i u_i^p(x) \geq \alpha_j u_j^p(y),$$

where α_i, $\alpha_j > 0$ satisfy $\alpha_i pz_i = \alpha_j pz_j$. By Theorem 2, we have $z \in LME(\succcurlyeq_E)$. If $\succcurlyeq_E$ were of the price-indicator type with respect to z and p, Theorem 2 would imply $z \in EIW(\succcurlyeq, p)$, that is, $pz_i = pz_j$, yielding a contradiction.

(iii) does not imply (i).

Let $z = (\Omega, 0, ..., 0)$, and let $p \in int\Delta^l$ be such that $z \in PO(\succcurlyeq, p)$. Consider the extended preference $\succcurlyeq_E^p$ from Example 1. Then, $z \notin EIW(\succcurlyeq, p)$.

(iii) does not imply (ii).

Using the same example, we have $(z_1, 1) \succ_E (z_i, i)$ for all $i \neq 1$. Hence, by Theorem 2, $z \notin LME(\succcurlyeq_E)$.

A necessary and sufficient condition for an extended preference to have a representation:

Fix an extended preference relation $\succcurlyeq_E$ throughout this subsection. We introduce the following property.

Continuity across agents: For any $i, j \in N$, and any sequences $x^{\nu}, y^{\nu} \in R^l_+ (\nu = 1, 2, ...,)$, if $(x^{\nu}, i) \succcurlyeq_E (y^{\nu}, j)$ for all ν and $x^{\nu} \longrightarrow x$, $y^{\nu} \longrightarrow y$, then $(x, i) \succcurlyeq_E (y, j)$.

This condition extends the usual notion of continuity for an individual's preference to comparisons involving different agents.

Proposition 1 *Suppose that $\succcurlyeq_E$ satisfies axiom E.1. Then $\succcurlyeq_E$ satisfies E.2 if and only if it exhibits continuity across agents.*

Proof. The "only if" direction is straightforward and therefore omitted. We now prove the "if" direction. The basic idea of the proof is that we regard the pair (x, i) as an extended consumption bundle, where i is treated as an additional good and consider a utility representation of $\succcurlyeq_E$. Let N be endowed with the discrete topology, and let $R^l_+ \times N$ be endowed with the product topology. By *continuity across agents,* $\succcurlyeq_E$ is continuous in the usual sense when regarded as a preference relation on $R^l_+ \times N$. As $R^l_+ \times N$ is second countable, there exists a continuous utility function v representing $\succcurlyeq_E$.[10] Thus, the desired representation $u = (u_i)_{i \in N}$ of $\succcurlyeq_E$ is given by the restriction of v: for any $i \in N$ and any $x \in R^l_+$, define $u_i(x) = v(x, i)$. This completes the proof. ■

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