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Hiromi Murakami and Ryo-Ichi Nagahisa

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関西大学ソシオネットワーク戦略研究機構

Research Institute for Socionetwork Strategies,
Kansai University
Joint Usage / Research Center, MEXT, Japan
Suita, Osaka, 564-8680, Japan
URL: https://www.kansai-u.ac.jp/riss/index.html
e-mail: riss@ml.kandai.jp
tel. 06-6368-1228
fax. 06-6330-3304

On the resource allocation problems with interpersonal comparisons of welfare: Extended preference approach revisited*

Hiromi Murakami† and Ryo-Ichi Nagahisa‡

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Abstract

This study introduces the extended preference approach studied in the literature of social welfare functionals into the pure exchange economy model. We clarify the relationship between equal income Walrasian (EIW) allocations and leximin equitable ones, showing that two axioms defined by the leximin criterion can characterize the EIW rule. We also discuss its implication to the problem of interpersonal comparisons of welfare.

Keywords: the equal income Walras rule; extended preference; the leximin criterion; interpersonal comparisons of welfare (ICW)

1 Introduction

The extended preference, a device used to make interpersonal comparisons of welfare (ICW) possible and studied in the field of Arrow's impossibility theorem, is applied to the resource allocation problem of exchange economies with a finite number of agents and goods.[1] At the same time, this study reveals a

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†Faculty of Fine Arts, Kyoto City University of Arts,13-6 Kutsukake-cho, Oe, Nishikyo-ku, Kyoto 610-1197, Japan Email: h_murakami@kcua.ac.jp

‡Department of Economics, Kansai University 3-3-35 Yamatecho Suita Osaka 564-8680 Japan E-mail: t940074@kansai-u.ac.jp

[1] d'Aspremont (1985), d'Aspremont and Gevers (2002), Sen (1970,1977,1986), and Suzumura (1983) surveyed this area. Fleurbaey and Hammond (2004) and Mongin and d'Aspremont (2004) investigated this field, considering a broader perspective on ethics and utility theories. Blackorby et al. (1984) provided a diagrammatic introduction. The most

few problems regarding ICW ignored in the previous research on social welfare functionals (SWLs).

We consider leximin fairness, a lexicographic sophistication of the difference principle of Rawls (1971) We show that the leximin equitable allocations defined on two specific types of extended preferences (price indicator type and locally created one) are equivalent to the equal income Walrasian (EIW) allocations. This result leads to a leximin axiomatization of the EIW rule.

An extended preference $\succsim_E$, defined on pairs of agent and consumption, can compare the welfare of different persons. Here, $(x, i) \succsim_E (y, j)$ implies that being agent i with consumption x is at least as well off as being agent j with consumption y. After defining the extended preference in the economic environments and discussing a few properties of those preferences, we investigate leximin equitable allocations, feasible allocations such that no other feasible ones are superior to those from the leximin point of view. We show the existence and characteristics of those allocations (Theorem 1).

Next, we study the relationship between the leximin equitable allocations and EIW ones, where two specific types of extended preferences play a crucial role. Theorem 2 shows that any two of the three concepts below imply the rest: the leximin equitable allocation, the EIW allocation, and the price indicator type of extended preferences. The third is one of the two specific types, which measures the welfare levels of different agents with a given price.

The other type is a locally created extended preference, made with only local information of subjective preferences, applying the idea of local independence (Nagahisa 1991, and Nagahisa and Suh 1995) to ICW. Theorem 3 shows that Theorem 2 still holds in most cases if replacing the price indicator type with a locally created one.

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recent contribution is Yamamura (2017).

The exceptions are Sen (1974a,b) and Deschamps and Gevers (1978), who considered the income distribution problem in a single-commodity economy.

The locally created extended preferences have more economic meaning regarding ICW than the price indicator type, whereas the latter is more mathematically tractable than the former in proofs.

The final goal of this study is the axiomatization of the EIW rule using the leximin criterion. We assume that a rule respects the spirit of the leximin fairness. Every allocation selected by it must meet the leximin criterion in some way. We define Leximin Justification Possibility (LMJP) and Leximin All Unanimity (LMAU). Take a feasible allocation arbitrarily. The former requires that if a rule selects it, at least one locally created extended preference must make the allocation leximin equitable. The latter says that if that holds for all the extended preferences, the rule must select it. We show that the EIW rule is the only rule meeting the two axioms (Theorem 4 and 5).

In addition to the importance of the leximin axiomatization, it is also worth noting the implication of Theorem 3, which says that if we hope that the set of EIW allocations is equal to the set of leximin equitable ones, we must use extended preferences locally created at those allocations. Thus, accepting the EIW rule while respecting the spirit of leximin fairness is equivalent to regarding interpersonal welfare comparisons according to the idea of locally created as correct, endorsing the validity of the suggestion proposed by Hammond (1991): The fact that people accept a rule is equivalent to their acknowledging the ICW based on that rule as correct. We show that a solution based on Hammond's strategy exists when focusing on exchange economies.

We address the interpersonal comparisons of welfare (ICW), not utilities (ICU). The cardinality of utilities is meaningless. The study uses utility representation of extended preferences (Proposition 1) in several places, which does not mean more than convenience, just as a typical usage of utility functions in microeconomic analysis.

The organization is as follows. Section 2 provides notation and definitions.

Section 3 examines a few properties of the leximin equitable allocations. Section 4 introduces the price indicator type and locally created one. After discussing a few properties of those preferences, we investigate the relationship between the leximin equitable allocations and the EIW ones using the two specific extended preferences. After formulating social choice rules and introducing axioms, Section 5 states the axiomatization of the EIW rule. Section 6 discusses the implication of the results, reviewing the previous research on the SWFLs and the idea suggested by Hammond (1991). Section 7 concludes. The appendix summarizes subordinate matters.

2 Notation and definitions

2.1 Exchange Economies

The notation for vector inequalities is $\geq$, $>$, and $\gg$. Let $\Delta^l := \{p \in R^l_+ : \sum_{i=1}^{l} p_i = 1\}$ and $int.\Delta^l := \{p \in R^l_{++} : \sum_{i=1}^{l} p_i = 1\}$.

We consider exchange economies with a finite number of agents and a finite number of private goods. Let $N = \{1, 2, ..., n\}$ and $L = \{1, 2, ..., l\}$ be the set of agents and the set of private goods respectively. All agents have the same consumption set R^l_+. Let $z_i = (z_{i1}, ..., z_{il}) \in R^l_+$ and $z = (z_1, ..., z_n) \in R^{nl}_+$ be agent i's consumption and an allocation respectively. Let $\Omega \in R^l_{++}$ be the total endowment of the economy, owned collectively and fixed throughout the study. An allocation z is feasible if $\sum_{i \in N} z_i \leq \Omega$. Let Z be the set of feasible allocations.

Let $\succcurlyeq_i$ be agent i's preference on R^l_+, where $\succ_i$ and $\sim_i$ read as usual. We assume that $\succcurlyeq_i$ is continuous, convex, and monotonic on R^l_+. We say that $\succcurlyeq_i$ is convex if for any $x, y \in R^l_+$, $x \succ_i y$ implies $tx + (1-t)y \succ_i y$ for all $t \in (0, 1)$ and that $\succcurlyeq_i$ is monotonic if $x > y$ implies $x \succ_i y$. Let Q be the set of preferences satisfying all the conditions. A list of all agents' preferences, denoted $\succcurlyeq = (\succcurlyeq_i)_{i \in N}$, is called a profile. Let $Q^n = Q \times \cdots \times Q$, where Q appears

n times, be the set of profiles.

Take a profile $\succsim$ arbitrarily. A feasible allocation z is Pareto optimal if and only if there are no other feasible ones z' with $z'_i \succ_i z_i$ for all $i \in N$.[2] Let $PO(\succsim)$ be the set of Pareto optimal allocations. Let $PO(\succsim, p)$ be the set of Pareto optimal allocations sharing with p as the supporting price vector. A feasible allocation z is an equal income Walrasian (EIW) allocation if there exists some $p \in int.\Delta^l$ such that $z_i \succsim_i x$ for all $x \in R^l_+$ with $px \leq p\left(\frac{\Omega}{n}\right)$. Let $EIW(\succsim)$ be the set of the EIW allocations. Let $EIW(\succsim, p)$ be a subset of $EIW(\succsim)$, where $p \in int.\Delta^l$ is an equilibrium price vector.

2.2 Extended Preferences

The notion of *extended preferences* is based on the principle of *extended sympathy* mentioned by Arrow (1963) and initiated by Suppes (1966) and Sen (1970). The basic idea is that a hypothetically existing ethical observer compares the welfare of different persons from a social point of view while respecting (or sympathizing with) their subjective preferences.

An extended preference $\succsim_E$ created from $\succsim \in Q^n$ is a complete and transitive binary relation on $R^l_+ \times N$. We read $(x, i) \succsim_E (y, j)$ as "being agent i with consumption x is at least as well off as being agent j with consumption y."[3] We read $\succ_E$ and $\sim_E$ as usual.

We say that a set of continuous utility functions $(u_i)_{i \in N}$ represents $\succsim_E$ if $(x, i) \succsim_E (y, j) \Longleftrightarrow u_i(x) \geq u_j(y)$ for all $i, j \in N$ and all $x, y \in R^l_+$. We call $(u_i)_{i \in N}$ a representation of $\succsim_E$.

We assume that extended preferences satisfy the following properties.

E.1. For any $i \in N$, $x \succsim_i y \Longleftrightarrow (x, i) \succsim_E (y, i)$ for all $x, y \in R^l_+$.

E.2. $(x, i) \succsim_E (0, j)$ for all $i, j \in N$ and all $x \in R^l_+$.

[2] The preferences are continuous and monotonic on R^l_+. Thus, the strong Pareto optimality reduces to the weak one.

[3] Note that we admit the comparability of individual welfare, but not the cardinality.

E.3. $\succcurlyeq_E$ has a representation.

E.1 is called the axiom of identity, assumed in almost all literature related to extended preferences with the abstract framework of social choice.[4] In contrast, E.2 is an assumption specific to the economic environment. It reflects our intuition that as long as other conditions are equal, people without wealth are the most miserable in the world. Note that assuming E.1, E.2 is equivalent to $(0, i) \sim_E (0, j)$ for all i, j. E.3 says that extended preferences should have their continuous utility representations. It is necessary just for technical reasons, as utility functions are so. The appendix shows the independence of the three properties and a necessary and sufficient condition for an extended preference satisfying E.1 and E.2 to have a representation.

Let $E(\succcurlyeq)$ be the set of extended preferences created from $\succcurlyeq$ satisfying the three properties. Note that $E(\succcurlyeq)$ is not empty: For each i, we define u_i representing $\succcurlyeq_i$ such that $u_i(0)$ is equal across all i. Then, comparing the utilities among agents makes an extended preference contained in $E(\succcurlyeq)$.

If we need to refer to multiple extended preferences, we use the notation $\succcurlyeq_{E'}$, $\succcurlyeq_{E''}$, and so on. Notice the difference between $\succcurlyeq_{E'}$ and $\succcurlyeq'_E$.

2.3 The leximin criterion

Let $\succcurlyeq_E$ be taken arbitrarily and fixed throughout the section. The leximin criterion, a lexicographic extension of the difference principle of Rawls (1971), is defined as follows. Given an allocation z, we arrange all (z_i, i) in ascending order such that $(z_{i_1}, i_1) \preccurlyeq_E (z_{i_2}, i_2) \preccurlyeq_E \cdots \preccurlyeq_E (z_{i_{n-1}}, i_{n-1}) \preccurlyeq_E (z_{i_n}, i_n)$, where tie is broken arbitrarily. The agent i_k $(k = 1, .., n)$ is the kth worst off agent in z. We denote i_k by $i_k(z)$. A lexicographic order $\geq_{L(E)}$ on the set of allocations is defined as follows:

[4] Refer to Sen (1970) and d'Aspremont (1985) for more details.

$$
\begin{array}{lcc}
 & & \exists k \in \{1,...,n\} \text{ s. t.} \\
z >_{L(E)} z' & \Longleftrightarrow & (z_{i_\tau(z)}, i_\tau(z)) \sim_E (z'_{i_\tau(z')}, i_\tau(z')) \ \forall \tau \in \{1,...,k-1\} \\
 & & \& \\
 & & (z_{i_k(z)}, i_k(z)) \succ_E (z'_{i_k(z')}, i_k(z')). \\
z =_{L(E)} z' & \Longleftrightarrow & (z_{i_k(z)}, i_k(z)) \sim_E (z'_{i_k(z')}, i_k(z')) \ \forall k = 1,...,n. \\
z \geq_{L(E)} z' & \Longleftrightarrow & z >_{L(E)} z' \vee z =_{L(E)} z'.
\end{array}
$$

Given z and z', z is at least as leximin just as z' if $z \geq_{L(E)} z'$. If this holds with $z >_{L(E)} z'$, z is more leximin just than z'. In contrast, if that does with $z =_{L(E)} z'$, z is equally as leximin just as z'. Note that all are well defined and transitive, and $\geq_{L(E)}$ is complete. A feasible allocation is leximin equitable if there is no other feasible one that is more leximin just than it. Let $LME(\succcurlyeq_E)$ be the set of those allocations. Note that $LME(\succcurlyeq_E) \subset PO(\succcurlyeq)$ holds because of E.1. An allocation z is leximin justifiable for $\succcurlyeq \in Q^n$ if there exists some $\succcurlyeq_E \in E(\succcurlyeq)$ such that $z \in LME(\succcurlyeq_E)$. The sequent section discusses the non-emptiness of $LME(\succcurlyeq_E)$ and a necessary and sufficient condition for an allocation to be leximin justifiable.

3 The properties of leximin equitable allocations

Take $\succcurlyeq_E$ arbitrarily. Let $EQ(\succcurlyeq_E)$ be defined as follows.

$$EQ(\succcurlyeq_E) = \{z \in Z : (z_i, i) \sim_E (z_j, j) \ \forall i, j \in N\}.$$

The theorem below assures the existence of leximin equitable allocations, characterizing the set of those allocations.

Theorem 1 *$\emptyset \neq LME(\succcurlyeq_E) = EQ(\succcurlyeq_E) \cap PO(\succcurlyeq)$ holds for any $\succcurlyeq \in Q^n$ and any $\succcurlyeq_E \in E(\succcurlyeq)$.*

Proof. Let $u = (u_i)_{i \in N}$ be a representation of $\succcurlyeq_E$, fixed throughout the proof. Note that E.3 assures existence. (i)-(iv) below complete the proof.

(i) $LME(\succsim_E) \subset EQ(\succsim_E) \cap PO(\succsim)$: As $LME(\succsim_E) \subset PO(\succsim)$, it suffices to show $LME(\succsim_E) \subset EQ(\succsim_E)$. Suppose not. Then, there exists some $z \in LME(\succsim_E)$ with $z \notin EQ(\succsim_E)$.

Let us arrange the pairs of agent and consumption at z in ascending order. We find that $<$ appears at some $k \in \{1, ..., n-1\}$ such that

$$u_{i_1}(z_{i_1}) \leq \cdots \leq u_{i_{k-1}}(z_{i_{k-1}}) \leq u_{i_k}(z_{i_k}) < u_{i_{k+1}}(z_{i_{k+1}}) \leq u_{i_{k+2}}(z_{i_{k+2}}) \leq \cdots \leq u_{i_n}(z_{i_n}).$$

Note that E.2 implies $z_{i_{k+1}} \neq 0$. Take $\varepsilon \in R^l_+/\{0\}$ sufficiently small such that the transfer ε of goods from agent i_{k+1} to agent i_k makes the ascending order unchanged. Note that this transfer is possible as u_{i_k} and $u_{i_{k+1}}$ are continuous functions. Let z' be the feasible allocation made from z by that transfer. Then, we have $z' >_{L(E)} z$, which contradicts $z \in LME(\succsim_E)$.

(ii) $EQ(\succsim_E) \cap PO(\succsim) \subset LME(\succsim_E)$: Suppose not. Then, there exists some $z \in EQ(\succsim_E) \cap PO(\succsim)$ with $z \notin LME(\succsim_E)$. Then, there exists some feasible allocation z' that is more leximin just than z. The diagram below illustrates the comparison of utilities in z and z'.

$$\begin{array}{ccccccccccc}
u_1(z_1) & = & \cdots & = & u_{k-1}(z_{k-1}) & = & u_k(z_k) & = & \cdots & = & u_n(z_n) \\
\shortparallel & & & & \shortparallel & & \wedge & & & & \\
u_{i_1}(z'_{i_1}) & \leq & \cdots & \leq & u_{i_{k-1}}(z'_{i_{k-1}}) & \leq & u_{i_k}(z'_{i_k}) & \leq & \cdots & \leq & u_{i_n}(z'_{i_n})
\end{array}$$

As z' is more leximin just than z, there exists k shown in the diagram. The diagram reveals $u_i(z'_i) \geq u_i(z_i)$ for all i, with inequality for $i_k, ..., i_n$, which contradicts $z \in PO(\succsim)$.

(iii) $\emptyset \neq LME(\succsim_E)$: The proof reduces to that of $EQ(u) \cap PO(u) \neq \emptyset$, where $EQ(u) = \{z \in Z : u_i(z_i) = u_j(z_j)\ \forall i, j\}$. Let U be a subset in utility space such that $U = \{(u_i(z_i))_{i \in N} : z = (z_i)_{i \in N} \in Z\}$. Imagine a 45-degree line on R^n passing through the origin. That line intersects with U at $(u_i(0))_{i \in N}$, as $u_i(0)$ is equal across all agents thanks to E.2. Noting U is closed and upper bounded, we can take the far upper-right intersection point among those intersection points.

We denote that point by $(u_i(z_i^*))_{i\in N}$. We show $z^* = (z_i^*)_{i\in N} \in EQ(u) \cap PO(u)$, which completes the proof.

As $z^* \in EQ(u)$ is by definition, the remaining is to prove $z^* \in PO(u)$. Suppose not. By noting the monotonicity of preferences, there exists some $z' \in Z$ such that $u_i(z_i') > u_i(z_i^*)$ for all i. Without loss of generality, we assume that agent n has the lowest utility in z'. Thus, for any $i \neq n$, $u_i(z_i') \geq u_n(z_n') > u_i(z_i^*)$. We can choose $\lambda_i \in [0,1]$ such that $u_i(\lambda_i z_i' + (1-\lambda_i) z_i^*) = u_n(z_n')$ for each i. Letting $z^\lambda \in Z$ be such that $\lambda_i z_i' + (1-\lambda_i) z_i^*$ for all i, we have $\left(u_i\left(z_i^\lambda\right)\right)_{i\in N} \in U$ and $u_i\left(z_i^\lambda\right) > u_i(z_i^*)$, which contradicts the remark above; $(u_i(z_i^*))_{i\in N}$ is the far upper-right intersection point among those intersection points. ■

The leximin criterion satisfies Hammond equity or its variants, which states that the smaller the disparity in utility between individuals, the more desirable.[5] It is generally impossible to eliminate those gaps when the alternatives are finite. However, under our assumption that goods are completely divisible and preferences are continuous, the utility disparities disappear, which is why leximin equitable allocations belong to $EQ(\succcurlyeq_E)$.

The existence of leximin equitable allocations is a welcome result. Moreover, its complete characterization is also impeccable. However, two unwelcome sides below result in, as a direct consequence of Theorem 1.

Corollary 1 *Any two allocations z and z' are leximin equitable for $\succcurlyeq_E$ if and only if they are identical on a preference basis, i.e., $z_i \sim_i z_i'$ for all i.*

Proof. A direct consequence of Theorem 1. ■

Corollary 2 *Let an allocation z be leximin justifiable for $\succcurlyeq \in Q^n$. Then, there exists $\succcurlyeq_{E'}$ such that $z \notin LME(\succcurlyeq_{E'})$.*

[5] Refer to Hammond 1976, d'Aspremont and Gevers 1977, and Deschamps and Gevers 1978.

Proof. By assumption, we can suppose $z \in LME(\succsim_E)$. Theorem 1 shows $(z_1, 1) \sim_E (z_2, 2) \sim_E \cdots \sim_E (z_n, n)$. Thus, if $z_i = 0$ for some i, E.2 implies $z = 0$, which contradicts the Pareto optimality of z. Thus, we have $z_i \neq 0$ for all i.

Let v_i be utility function representing $\succsim_i$ such that

$$v_i(0) = 0 \text{ for all } i, \text{ and } v_1(z_1) = v_2(z_2) = \cdots = v_{n-1}(z_{n-1}) < v_n(z_n),$$

which is possible because of $z_i \neq 0$ for all i. Let $\succsim_{E'}$ be the extended preference making from $(v_i)_{i \in N}$; $(v_i)_{i \in N}$ is a representation of $\succsim_{E'}$. Invoking Lemma 1, we have $z \notin LME(\succsim_{E'})$, a desired result. ■

These results pose difficulty in axiomatizing the EIW rule using the leximin criterion. In most cases, a single extended preference cannot make all EIW allocations leximin justifiable (Corollary 1). Thus, we have no choice but to use multiple ones. However, these will likely create contradictory leximin judgments (Corollary 2). We will deal with this issue later.

4 Two types of extended preferences

This section introduces two specific types of extended preferences relating leximin equitable allocations with EIW ones.

Suppose $z \in PO(\succsim, p)$. An extended preference $\succsim_E$ is a *price indicator type* at z and p if for any $i, j \in N$ and any $x \in R^l_+$,

$$(x, j) \succsim_E (z_i, i) \text{ implies } px \geq pz_i, \text{ and } (x, j) \succ_E (z_i, i) \text{ implies } px > pz_i.$$

Note that $(z_i, i) \succsim_E (z_j, j)$ holds if and only if $pz_i \geq pz_j$. The price indicator type compares the welfare of different agents at z using p, judging that the richer, the better, where we interpret pz_i as the wealth of agent i.

Remark 1 *The price indicator type is well-defined for any* $z \in PO(\succsim, p)$*. We invoke money-metric utility functions (Arrow and Hahn 1971 p106), formulated as* $u_i^p(x) = \min\{pq : q \sim_i x\}$[6]*. A price indicator type* $\succsim_E^p$ *at* z *and* p *is such that for any* $x, y \in R_+^l$ *and any* $i, j \in N$*,*

$$(x, i) \succsim_E^p (y, j) \Longleftrightarrow u_i^p(x) \geq u_j^p(y).$$

Note that $\succsim_E^p$ *meets E.1-3.*

Theorem 2 below shows that the price indicator type relates leximin equitable allocations with EIW allocations.

Theorem 2 *Take* $z \in PO(\succsim, p)$ *and* $\succsim_E \in E(\succsim)$ *arbitrarily. Any two of the three below imply the rest.*

(i) $z \in EIW(\succsim, p)$*.*

(ii) $z \in LME(\succsim_E)$*.*

(iii) $\succsim_E$ *is a price indicator type at* p *and* z*.*

Proof. (i) and (ii) imply (iii): We show that $(x, j) \succsim_E (z_i, i)$ implies $px \geq pz_i$. Suppose $(x, j) \succsim_E (z_i, i)$. As Theorem 1 and (ii) show $(z_i, i) \sim_E (z_j, j)$ for all i, j, we have $(x, j) \succsim_E (z_j, j)$, which implies $x \succsim_j z_j$ because of E.1. Thus, (i) implies $px \geq pz_j$, a desired result. We can show that $(x, j) \succ_E (z_i, i)$ implies $px > pz_j$ as well, which completes the proof.

(ii) and (iii) imply (i): Theorem 1 and (ii) show $(z_i, i) \sim_E (z_j, j)$ for all i, j. As $\succsim_E$ is the price indicator type at p and z, this implies $pz_i = pz_j$ for all i, j. Noting $z \in PO(\succsim, p)$, we have $z \in EIW(\succsim)$, the desired result.

(iii) and (i) imply (ii): Suppose $(z_j, j) \succ_E (z_i, i)$ for some i, j. By (iii), this implies $pz_j > pz_i$, which contradicts $pz_i = pz_j$, a direct consequence of (i). Thus, we complete $z \in EQ(\succsim_E)$. Noting $z \in PO(\succsim)$, Theorem 1 implies $z \in LME(\succsim_E)$, a desired result. ■

[6] Refer to Weymark (1985) for details of this function.

Do not misunderstand that Theorem 2 implies the equivalence of (i)-(iii). What it implies is (i)+(ii)=(ii)+(iii)=(iii)+(i). Note that (i)-(iii) are logically independent. Corollary 1 shows that (i) alone does not necessarily imply (ii). Similarly, (i) alone does not necessarily imply (iii) as $E(\succcurlyeq)$ has a rich structure. Refer to the Appendix for the other cases.

Theorem 2 says the following.

(i) and (ii) imply (iii): If an extended preference makes an EIW allocation leximin equitable, it must be a price indicator type.

(ii) and (iii) imply (i): If a feasible allocation is leximin justifiable for an extended preference, which is a price indicator type, it is an EIW allocation.

(iii) and (i) imply (ii): For any EIW allocation, if we make an extended preference, which is a price indicator type using the associated equilibrium prices, that allocation is leximin justifiable for that preference.

Next, we consider extended preferences created with only local information. The meaning of locally creating is similar to local independence (Nagahisa 1991, and Nagahisa and Suh 1995). The formal definition is as follows.

Let $p_N = (p_i)_{i \in N}$, where $p_i \in int.\Delta^l$, and $z \in Z$ be given. We say that $\succcurlyeq$ and $\succcurlyeq'$ are p_N identical at z if for any i, $x \succcurlyeq_i z_i$ implies $p_i x_i \geq p_i z_i$ and $x \succcurlyeq'_i z_i$ implies $p_i x_i \geq p_i z_i$. Here, p_i is a common supporting price vector for $\succcurlyeq_i$ and $\succcurlyeq'_i$. Let $p_N(\succcurlyeq, z)$ be the set of profiles that are p_N identical to $\succcurlyeq$ at z. If $\succcurlyeq$ and $\succcurlyeq'$ are p_N identical at z, then $p_N(\succcurlyeq, z) = p_N(\succcurlyeq', z)$. Thus, we write $p_N(\cdot, z)$ if there is no need to specify $\succcurlyeq$.

The welfare ranking of $\succcurlyeq_E$ at z, denoted by $\geq_{(\succcurlyeq_E, z)}$, is the order on N such that $i \geq_{(\succcurlyeq_E, z)} j \iff (z_i, i) \succcurlyeq_E (z_j, j)$ for all i, j, which is a list that indicates who is the wealthiest at z, who is the second, ..., and who is the worse off at z.

Now, we define

$$\bigcap_{\succcurlyeq \in p_N(\cdot,z)} \left\{ \geq_{(\succcurlyeq_E,z)} : \succcurlyeq_E \in E(\succcurlyeq) \right\} \tag{1}$$

or equivalently

$$\bigcap_{\succcurlyeq' \in p_N(\succcurlyeq,z)} \left\{ \geq_{(\succcurlyeq'_E,z)} : \succcurlyeq'_E \in E(\succcurlyeq') \right\}.$$

We say that an extended preference $\succcurlyeq_E$ is locally created at p_N and z if and only if $\geq_{(\succcurlyeq_E,z)}$ belongs to (1). Then, $\geq_{(\succcurlyeq_E,z)}$, the welfare ranking at z, is determined only by preferences around z, not depending on global information far away from z. To simplify the discussion, we assume that individual preferences are smooth and z is an interior point. Then, p_N matches marginal rates of substitution (MRS) at z, equaling across all $\succcurlyeq' \in p_N(\succcurlyeq, z)$, which are therefore approximately identical to each other when we focus on only around z. If MRSs at z are the only information available to create extended preferences, they have the same information for deciding the welfare ranking at z. Thus, we conclude that the set of welfare rankings at z is equal across all $\succcurlyeq' \in p_N(\succcurlyeq, z)$. As a result, the welfare rankings made from a locally created extended preference belong to (1).

If there is no need to mention p_N, we call $\succcurlyeq_E$ an extended preference locally created at z. If z also does not matter, we omit "at z" either. As we assume neither smoothness nor interiority, different supporting prices $p'_N = (p'_i)_{i \in N}$ may define different sets (1) for the same profile $\succcurlyeq$. If preferences are assumed to be smooth, this diversity disappears. We will also discuss this case later.

The locally created extended preferences remind us of local independence (LI) (Nagahisa 1991, Nagahisa and Suh 1995), used to characterize the Walras rule. LI is a choice consistency requirement of social choice: It requires the decision to choose z to be the same between $\succcurlyeq$ and $\succcurlyeq'$ if MRS equal condition holds for z. On the other hand, the idea of locally created extended preferences does not require that much. It only demands the information about welfare ranking at z to remain unchanged, which does not necessarily imply that the decision to choose z must be the same.

From now on, all discussions need the following additional assumption regarding $E(\succcurlyeq)$. If every agent has the same preference, we write $\succcurlyeq = (\succcurlyeq, ..., \succcurlyeq)$, permitting abuse of notation. Let $\succcurlyeq_{E^*}$ be the extended preference such that for any $i, j \in N$ and any $x, y \in R^l_+$, $(x, i) \succcurlyeq_{E^*} (y, j) \Leftrightarrow x \succcurlyeq y$. We modify $E(\succcurlyeq)$ as follows.

D.1. If every agent has the same preference, then $E(\succcurlyeq) = \{\succcurlyeq_{E^*}\}$.[7]

There is no compelling reason to dismiss D.1. If everyone has the same preference, that preference should be the only extended preference, and no other extended one is considered possible. We leave $E(\succcurlyeq)$ unchanged except for this peculiar case.

Now, we prove two lemmas for locally created extended preferences.

Lemma 1 *Suppose* $z \in PO(\succcurlyeq, p)$ *and* $pz_1 \leq pz_2 \leq \cdots \leq pz_n$ *without loss of generality. Then, locally created extended preferences exist at* p *and* z*, all of which make the same welfare ranking at* z *as* $pz_1 \leq pz_2 \leq \cdots \leq pz_n$*, that is*

$$pz_1 \leq pz_2 \leq \cdots \leq pz_n \Leftrightarrow (z_1, 1) \preccurlyeq_E (z_2, 2) \preccurlyeq_E \cdots \preccurlyeq_E (z_n, n). \tag{2}$$

Proof. To simplify the notation, we use $\geq^p$ as the welfare ranking of (2). Let us show

$$\{\geq^p\} = \overbrace{\bigcap_{\succcurlyeq' \in p(\succcurlyeq, z)} \left\{\geq_{(\succcurlyeq'_E, z)} : \succcurlyeq'_E \in E(\succcurlyeq')\right\}}^{\text{the set of (1)}}. \tag{3}$$

(3) completes the proof. First, we show the following.

$$\geq^p \in \overbrace{\bigcap_{\succcurlyeq' \in p(\succcurlyeq, z)} \left\{\geq_{(\succcurlyeq'_E, z)} : \succcurlyeq'_E \in E(\succcurlyeq')\right\}}^{\text{the set of (1)}}. \tag{4}$$

[7] Note that $\succcurlyeq_{E^*}$ meets E.1-3. Note also that D.1 is not an assumption on extended preference but the set of those preferences.

Take $\succsim' \in p(\succsim, z)$ arbitrarily. Let $\succsim'_{E^*} \in E(\succsim')$ be an extended preference defined by money-metric utility functions made from using p (Refer to Remark 1). Then, $\succsim'_{E^*}$ makes the same welfare ranking as $\geq^p$ at z. Thus, we have

$$\geq^p \in \left\{ \geq_{(\succsim'_E, z)} : \succsim'_E \in E(\succsim') \right\}$$

As this holds for all $\succsim' \in p(\succsim, z)$, we have (4).

Next, let $\succsim^p$ be a profile such that every agent has the same preference represented by the utility function $u(x) = px$. Because of D.1, $\succsim^p$ itself is the only possible extended preference for $\succsim^p$, and the welfare ranking at z is the same as $\geq^p$. Thus, we have

$$\left\{ \geq_{(\succsim^p_E, z)} : \succsim^p_E \in E(\succsim^p) \right\} = \{\geq^p\} \tag{5}$$

By noting $\overbrace{\bigcap_{\succsim' \in p(\succsim, z)} \left\{ \geq_{(\succsim'_E, z)} : \succsim'_E \in E(\succsim') \right\}}^{\text{the set of (1)}} \subset \left\{ \geq_{(\succsim^p_E, z)} : \succsim^p_E \in E(\succsim^p) \right\}$, (4) and (5) complete (3). ■

Lemma 2 *Suppose $z \in EIW(\succsim, p)$. The three statements below are equivalent to each other.*

(i) $\succsim_E \in E(\succsim)$ is locally created at p and z.

(ii) $\geq_{(\succsim_E, z)}$ is to weigh every agent well off equally: $(z_1, 1) \sim_E (z_2, 2) \sim_E \cdots \sim_E (z_n, n)$.

(iii) $\succsim_E \in E(\succsim)$ is a price indicator type at p and z.

Proof. As (ii)$\Longleftrightarrow$(iii) is straightforward, the only remaining is (i)$\Longleftrightarrow$(ii). Let $\geq^p$ be the welfare ranking at z, which is the same as (ii). Following the same procedure as Lemma 1, we have

$$\{\geq^p\} = \overbrace{\bigcap_{\succsim' \in p(\succsim, z)} \left\{ \geq_{(\succsim'_E, z)} : \succsim'_E \in E(\succsim') \right\}}^{\text{the set of (1)}}. \tag{6}$$

(6) completes (i)$\Longleftrightarrow$(ii). ■

Replacing "a price indicator type" in Theorem 2 with "locally created," we obtain Theorem 3 below.

Theorem 3 *Take $z \in PO(\succcurlyeq, p)$ and $\succcurlyeq_E \in E(\succcurlyeq)$ arbitrarily. Any two of the three below imply the rest.*

(i) $z \in EIW(\succcurlyeq, p)$.

(ii) $z \in LME(\succcurlyeq_E)$.

(iii) $\succcurlyeq_E$ is locally created at p and z.

Proof. As Theorem 2 and Lemma 2 imply that (i)+(ii)$\Longrightarrow$(iii) and (iii)+(i)$\Longrightarrow$(ii), the only remaining is (ii)+(iii)$\Longrightarrow$(i). Theorem 1 and (ii) imply $(z_1, 1) \sim_E (z_2, 2) \sim_E \cdots \sim_E (z_n, n)$, which together with Lemma 1 shows $pz_1 = pz_2 = \cdots = pz_n$, a desired result. ■

5 Rules

A rule F is a mapping that associates with each profile $\succcurlyeq \in Q^n$ a nonempty subset of Z. However, it decides $F(\succcurlyeq)$, comparing the welfare of different agents. Let $\mathcal{D} = \bigcup_{\succcurlyeq \in Q^n} E(\succcurlyeq)$ be the extended domain, interpreted as the set of extended preferences used for social decisions. The figure below illustrates F:

$$\succcurlyeq \quad \overset{\text{Extension}}{\longrightarrow} \quad E(\succcurlyeq) = \{\succcurlyeq_E, \succcurlyeq_{E'}, ...\} \quad \overset{\text{Axioms}}{\longrightarrow} \quad F(\succcurlyeq)$$

Rule F associates with each profile $\succcurlyeq$ a nonempty set of feasible allocations $F(\succcurlyeq)$, using extended preferences $\succcurlyeq_E, \succcurlyeq_{E'}$ created from $\succcurlyeq$.

Part $\overset{\text{Extension}}{\longrightarrow}$ is the remarkable difference from existing studies in the literature. A profile can create multiple (sometimes infinite) extended preferences. However, not all of them deserve consideration in social decisions. For example, no ethical or social grounds exist for a judgment that being a beggar is better than being a billionaire. Comparing the welfare of different persons is the core of the issue, having been avoided in the study on social welfare functionals (SWFLs), which define F as $\succcurlyeq_E \overset{\text{Axioms}}{\longrightarrow} F(\succcurlyeq_E)$ and permit any extended preferences to be used for social decisions. We will put aside the reasons for the avoidance later, only stressing two points here. First, when we claim that the welfare of individual i is more valuable than that of j, it must have plausible grounds with no strong value judgments, though considered an intricate problem in the history of welfare economics. Second, even if succeeding in solving this problem, another problem remains: we cannot select only one extended preference needed for social decisions, as Corollary 1 shows that, in most cases, all allocations in $F(\succcurlyeq)$ cannot be leximin equitable with only one extended preference. In contrast, Corollary 2 demonstrates that if multiple ones considered equally plausible remain, some can yield contradictable judgments. Thus, the problem is what judgments we should adopt and give priority.

The two devices already discussed define appropriate extended preferences used for social decisions. The first (D.1) is a self-evident truth, requiring that if every agent has the same preference, that preference should be the only extended preference. The second is the concept of locally created extended preference, a requirement of informational economization in making ICW, which requires the welfare ranking at an allocation to be determined only by local information of individual preferences around the allocation, independently from preferences about allocations far away. Those procedures modify $E(\succcurlyeq) = \{\succcurlyeq_E, \succcurlyeq_{E'}, ...\}$ and select appropriate extended preferences from it.

After defining extended preferences used for social decisions this way, we proceed to consider the way to synthesize multiple leximin judgments created from those preferences. That is the part of $\overset{\text{Axioms}}{\longrightarrow}$. Rules must conform to the spirit of the leximin criterion; every allocation selected must be leximin equitable in some way. There may be various ways of judging, but at least we must observe the following two principles.

1. If we want to select z, at least one extended preference must make it leximin equitable. Otherwise, we would always do so regardless of the leximin criterion.

2. If z becomes leximin equitable for all extended preferences, then we must select it. Otherwise, z would never be selectable, and the leximin criterion would be of no use.

Leximin Justification Possibility (LMJP) crystallizes the first idea, and Leximin All Unanimity (LMAU) embodies the second. A rule F satisfies LMJP if for any $z \in F(\succsim)$, there exists some $\succsim_E \in E(\succsim)$ that is locally created at z and $z \in LME(\succsim_E)$. A rule F meets LMAU if $z \in F(\succsim)$ holds whenever $z \in LME(\succsim_E)$ is true for all $\succsim_E \in E(\succsim)$ that are locally created at z. Both are the most appropriate ways to integrate leximin judgments in cases where one judgment is insufficient, but multiple judgments may contradict each other.

Let $LC(\succsim, z)$, a subset of $E(\succsim)$, consist of extended preferences that are locally created at z. Let

$$LME^{\exists}(\succsim) = \left\{ z \in Z : z \in \bigcup_{\succsim_E \in LC(\succsim, z)} LME(\succsim_E) \right\}$$
$$LME^{\forall}(\succsim) = \left\{ z \in Z : z \in \bigcap_{\succsim_E \in LC(\succsim, z)} LME(\succsim_E) \right\}.$$

Using those notations, we can make more concise definitions of LMJP and LMAU.

$$\begin{array}{lcl} \text{LMJP} & : & F(\succsim) \subset LME^{\exists}(\succsim) \text{ for all } \succsim \in Q^n. \\ \text{LMAU} & : & LME^{\forall}(\succsim) \subset F(\succsim) \text{ for all } \succsim \in Q^n. \end{array}$$

Theorem 1 and Remark 1 assure that $LME^{\exists}(\succsim)$ is nonempty. Theorem 4 below ensures that $LME^{\forall}(\succsim)$ is nonempty, strengthening so much the equivalence relation between EIW allocations and leximin equitable ones clarified by Theorem 3.

Theorem 4 *$LME^{\forall}(\succsim) = EIW(\succsim) = LME^{\exists}(\succsim)$ for all $\succsim \in Q^n$.*

Proof. The (i)-(iii) below completes the proof.

(i) $LME^{\forall}(\succsim) \subset EIW(\succsim)$: Suppose not. Then, there exists $z \in Z$ such that $z \in \bigcap_{\succsim_E \in LC(\succsim, z)} LME(\succsim_E)$ and $z \notin EIW(\succsim)$. Let $z \in PO(\succsim, p)$, and the ordering be $pz_1 \geq pz_2 \geq \cdots \geq pz_n$ without loss of generality. Lemma 1 implies that there exists $\succsim_E \in LC(\succsim, z)$ such that $\geq_{(\succsim_E, z)}$is the same as $pz_1 \geq pz_2 \geq \cdots \geq pz_n$. Here, noting $z \notin EIW(\succsim)$, $\geq$ holds with strict relation somewhere in the ordering, which contradicts $z \in LME(\succsim_E)$, considering Theorem 1.

(ii) $EIW(\succsim) \subset LME^{\exists}(\succsim)$: Take $z \in EIW(\succsim, p)$. Take $\succsim_E \in E(\succsim)$, a price indicator type at p and z, arbitrarily. Refer to Remark 1 for the existence. Then, Theorem 2 and 3 show that $z \in LME(\succsim_E)$ and $\succsim_E \in LC(\succsim, z)$, which implies $z \in LME^{\exists}(\succsim)$, a desired result.

(iii) $LME^{\exists}(\succsim) \subset LME^{\forall}(\succsim)$: Take $z \in LME^{\exists}(\succsim)$. There exists $\succsim_E \in LC(\succsim, z)$ with $z \in LME(\succsim_E)$. Let $p \in int.\Delta^l$ be a supporting price vector associated with z. As $\succsim_E \in LC(\succsim, z)$, $\geq_{(\succsim_E, z)}$ is contained in $\overbrace{\bigcap_{\succsim' \in p(\succsim, z)} \left\{ \geq_{(\succsim'_E, z)} : \succsim'_E \in E(\succsim) \right\}}^{\text{the set of (1)}}$. Let $\succsim^p$ be a profile where every agent has the same preference represented by the utility function $u(x) = px$. Then, we have

$$\geq_{(\succsim_E, z)} \in \overbrace{\bigcap_{\succsim' \in p(\succsim, z)} \left\{ \geq_{(\succsim'_E, z)} : \succsim'_E \in E(\succsim) \right\}}^{\text{the set of (1)}} \subset \left\{ \geq_{(\succsim^p_E, z)} : \succsim^p_E \in E(\succsim^p) \right\}.$$

Because of D.1, the right-hand set is singleton. Thus, we have

$$\left\{ \geq_{(\succsim_E, z)} \right\} = \overbrace{\bigcap_{\succsim' \in p(\succsim, z)} \left\{ \geq_{(\succsim'_E, z)} : \succsim'_E \in E(\succsim) \right\}}^{\text{the set of (1)}}.$$

The above relation implies that every $\succsim_{E'} \in LC(\succsim, z)$ creates the same welfare ranking at z, and the same as $\geq_{(\succsim_E, z)}$. Thus, Theorem 1 implies $z \in \bigcap_{\succsim_E \in LC(\succsim, z)} LME(\succsim_E)$, a desired result. ■

Now, we state the leximin axiomatization of the EIW rule, which is a direct consequence of Theorem 4, so omit the proof.

Theorem 5 *The EIW rule is the only rule satisfying LMJP and LMAU.*

The independence of the axioms is easy to show. Any strict subcorrespondence of the EIW rule satisfies LMJP but not LMAU. All select rule, selecting all feasible allocations for any case, meets LMAU but not LMJP.

Let us consider the case of smoothness of preferences. Let $\widehat{Q}$ be a subset of Q consisting of smooth preferences subject to a boundary condition such that $\left\{0 \leq x \leq \Omega : x \succcurlyeq \frac{\Omega}{n}\right\} \subset R^l_{++}$.The boundary conditions assure the interiority of EIW allocations. A rule F is a mapping that associates with each profile $\succcurlyeq \in \widehat{Q}^n$ a nonempty subset of Z. In addition to E.1-3, the extended domain meets the following:

E.4 For any $\succcurlyeq_E$, there exists $z \in Z$ such that $(z_1, 1) \sim_E (z_2, 2) \sim_E \cdots \sim_E (z_n, n)$ and $\{0 \leq x \leq \Omega : x \succcurlyeq_i z_i\} \subset R^l_{++}$. for all i.

Considering Theorem 1, E.4 assures every leximin equitable allocation is in the interior: $LME(\succcurlyeq_E) \subset R^{nl}_{++}$ for all $\succcurlyeq_E \in E(\succcurlyeq)$. Let us reinterpret $E(\succcurlyeq)$ as the set of extended preferences meeting E.1-E-4. Note that D.1 still holds under this reinterpretation. Remark 1 assures the existence of such extended preferences.

The results survive under this slight modification. The only necessity in proofs is replacing $\succcurlyeq^p$ with a profile represented by the same CES utility function, choosing parameters appropriately to fit boundary conditions.

6 Discussion

We discuss two issues here. The first is the relevance of the previous research on SWFLs. No literature on SWFLs addressed the problem of which extended preferences we use for social decisions. They were only concerned about the axiomatic comparison of the leximin and utilitarian rules defined in the SWFL framework and the Arrovian ones in the traditional approach. In particular, they focused on the characterization by using invariance axioms, which classify utilities and preferences from an informational point of view. For this purpose, the problem of considering the grounds on which we give priority to some extended preferences to others, the main issue we addressed in this study, makes the comparison unclear.

Second, Theorem 3 is especially valuable, having three implications. Of these, "if an EIW allocation is leximin justifiable, then that extended preference is a locally created one" is the most remarkable. The message is that if we wish to value the equal income Walras rule and the leximin criterion, we must only use the locally created extended preferences. In connection with this argument, Hammond (1991) stated a thought-provoking idea about interpersonal comparisons of utilities (ICUs):

But if it is easier to think what is a good social welfare ordering, rather than how to make ICUs, why should we not start with the ordering and have it reveal the ICUs, instead of starting with ICUs and trying to derive a social ordering? Especially if it is not at all clear anyway how to incorporate ICUs into a social ordering even if we believe we have made securely founded and ethically relevant interpersonal comparisons of both utility levels and utility differences.

-Hammond (1991, p226-227)

This study reveals one relationship, which Hammond remarked "not at all clear," between ICUs and resource allocation rules. Though only dealing with exchange economies, thus a limited argument, our result answers Hammond's question. The fact that people accepted the rules corresponds to the fact that people agreed with making welfare comparisons based on the rules because each person receives their utility according to the rules. In this sense, we are making welfare comparisons between individuals, as discussed elsewhere in Hammond (1991).[8] Accepting the EIW rule and the leximin fairness as valuable is equivalent to justifying only locally created extended preferences or equivalently regarding welfare comparisons by those extended preferences as correct.

7 Conclusion

This paper combines two studies advanced independently in social choice. One is the study of interpersonal comparisons of welfare, and the other is the study of axiomatic analysis of resource allocation problems.[9] Several studies have proved the advantages of the EIW rule from normative points of view.[10] This paper also belongs to that stream of research.

In this study, we came across the question of which extended preferences we should use for social decisions, overlooked in previous research on SWFLs. This issue is closely related to Hammond's idea on ICU, and further research is worth doing in this direction.

8 Appendix

The independence of E.1-E.3:

E.1: Take a profile $\succcurlyeq$ such that agent 1's preference is not representable by linear utility functions. Let $\succcurlyeq_E$ be such that there exists $p \in int.\Delta^l$ such that $(x, i) \succcurlyeq_E (y, j) \Longleftrightarrow px \geq py$. Then, $\succcurlyeq_E$ satisfies E.2 and E.3, but not E.1.

E.2: Let u_i be agent i's continuous utility function representing $\succcurlyeq_i$ and $u_1(0) > u_2(0)$. Let $\succcurlyeq_E$ be the extended preference made by utility comparison using the u_is. Then, $\succcurlyeq_E$ satisfies E.1 and E.3, but not E.2.

[8] Refer to Section 6.3 of Hammond (1991) for more details.

[9] Chamber and Hayashi (2017) proposed an alternative axiomatization of the Walras rule considering income distribution problems.

[10] Thomson (2007) includes a comprehensive survey on the justifications with no envy concept. Fleurbaey and Maniquet (2011) study the normative aspect of the equal income Walras rule with social ordering functions, a rational choice function version of Arrow's social welfare function.

E.3: For any consumption x, let $S(x) = \sum_{h=1}^{l} x_h$. Let utility functions be such that

$$\begin{aligned} u_j(x) &= 1 - \frac{1}{1+S(x)}, \\ u_k(x) &= 2 - \frac{1}{1+S(x)}, \text{ and} \\ u_i(x) &= S(x) \text{ for all } i \neq j, k. \end{aligned}$$

Let $\succsim$ be the profile induced from those utility functions. Now we change agent k's utility function in the following:

$$\widetilde{u}_k(x) = \begin{cases} 0 & \text{if } x = 0 \\ 2 - \frac{1}{1+S(x)} & \text{otherwise} \end{cases}$$

Let $\succsim_E$ be the extended preference made by utility comparison using $\widetilde{u}_k$, u_j, and u_i $(i \neq j, k)$. Then, $\succsim_E$ satisfies E.1 and E.2, but not E.3; Suppose, on the contrary, that $\succsim_E$ has a representation $(v_i)_{i \in N}$. By definition, we have $v_j(x) < v_k(y)$ for all $x, y \neq 0$. Letting $y \longrightarrow 0$, we have $v_j(x) \leq v_k(0)$ for all $x \neq 0$, and hence $v_j(0) < v_k(0)$, which implies $(0, j) \prec_E (0, k)$, a contradiction.

A necessary and sufficient condition for an extended preference to have a representation: We propose two properties below.

Continuity among agents: For any $i, j \in N$, and any $x^\nu, y^\nu \in R^l_+ (\nu = 1, 2, ...,)$, if $(x^\nu, i) \succsim_E (y^\nu, j)$ for all ν and $x^\nu \longrightarrow x$, $y^\nu \longrightarrow y$, then $(x, i) \succsim_E (y, j)$.

Non-satiation among agents: For any $(x, i) \in R^l_+ \times N$, and any $j \in N$, there is some $y \in R^l_+$ such that $(x, i) \prec_E (y, j)$.

These are conditions that make the continuity and non-satiation of a preference of a single agent applicable between different agents.

Proposition 1 *An extended preference having E.1 and E.2 meets E.3 if and* *only if it satisfies continuity and non-satiation among agents.*

Proof. If: Let $\succcurlyeq_E$ be an extended preference. Let e be a consumption such that $e = (1, 1, ..., 1)$, where 1 repeats l times. Let $I = \{te \in R^l_+ : t \geq 0\}$. We define u_1 by $u_1(x) = t_x$, where $t_x e \sim_1 x$. The proof goes on with two steps.

Step 1. For any $(x, i) \in R^l_+ \times N$, $i \neq 1$, there is a unique $t \geq 0$ such that $(x, i) \sim_E (te, 1)$.

If it is not unique, then $(x, i) \sim_E (te, 1)$ and $(x, i) \sim_E (t'e, 1)$ holds for $t \neq t'$. Thus, $(te, 1) \sim_E (t'e, 1)$, and hence $te \sim_1 t'e$ because of E.1, which contradicts the monotonicity of preferences.

Next, we show the existence. If $x = 0$, then E.2 implies that $t = 0$ is the desired one. Suppose $x \neq 0$. Then, we have $(0,1) \overset{\text{E.2}}{\overset{\downarrow}{\sim_E}} (0, i) \overset{\text{monotonicity}}{\overset{\downarrow}{\prec_E}} (x, i)$. Non-satiation between agents means that there is some $z \in R^l_+$ such that $(x, i) \prec_E (z, 1)$. On the other hand there exists a unique $\widehat{te} \in I$ such that $(\widehat{te}, 1) \sim_E (z, 1)$. We conclude $(0, 1) \prec_E (x, i) \prec_E (\widehat{te}, 1)$. Take a segment $[0, \widehat{te}]$. If there is no te on the segment such that $(te, 1) \sim_E (x, i)$, then the segment is divided into two nonempty open sets,$\{te \in [0, \widehat{te}] : (te, 1) \prec_E (x, i)\}$ and $\{te \in [0, \widehat{te}] : (x, i) \prec_E (te, 1)\}$, which contradicts the connectedness of the segment.[11] Hence, we have $te \in I$ with $(x, i) \sim_E (te, 1)$, which completes the proof of Step 1.

Step 2. We completes the proof.

For each $i \neq 1$, we define a continuous mapping $\varphi_{i1} : R^l_+ \longrightarrow R^l_+$ that associates each $x \in R^l_+$ with $\varphi_{i1}(x) = te$ such that $(x, i) \sim_E (te, 1)$.[12]

The desired utility functions are u_1 and $u_i := u_1(\varphi_{i1})$, $(i = 2, ..., n)$. The arrows below completes the proof:

$$(x, i) \succcurlyeq_E (y, j) \overset{\text{def. of } \varphi_{i1} \text{ and } \varphi_{j1}}{\overset{\downarrow}{\Longleftrightarrow}} (\varphi_{i1}(x), 1) \succcurlyeq_E (\varphi_{j1}(y), 1) \overset{\text{E.1}}{\overset{\downarrow}{\Longleftrightarrow}} \varphi_{i1}(x) \succcurlyeq_1$$

[11] The openness follows from the continuity among agents. The nonemptyness follows from the fact that 0 belongs to the first set and $\widehat{te}$ belongs to the second set.

[12] The continuity of φ_{i1} follows from the continuity among agents.

$\varphi_{j1}(y)$

$$\Longleftrightarrow u_1(\varphi_{i1}(x)) \geq u_1(\varphi_{j1}(y)) \quad \overset{\text{def. of } u_i \text{ and } u_j}{\overset{\downarrow}{\Longleftrightarrow}} \quad u_i(x) \geq u_j(y),$$

where we set $u_1 = u_1(\varphi_{11})$ and φ_{11} is identity mapping.

Only if: As continuity is obvious, we show non-satiation. Suppose, on the contrary, that there exist agents i and j, and a consumption y^0 such that $u_i(x) < u_j(y^0)$ for all $x \in R^l_+$. Let $I = \{u_i(x) : x \in R^l_+\}$ and $J = \{u_j(y) : u_i(x) < u_j(y)$ for all $x \in R^l_+\}$. Consider $\sup I$ and $\inf J$. As I is upper-bounded and J is lower-bounded, $\sup I$ and $\inf J$ can be well-defined.

We show that there exists some $y^* \in R^l_+$ such that $u_j(y^*) = \inf J$. Take some $y \in J$ arbitrarily. Take the segment between 0 and y. By noting $u_i(0) = u_j(0) < u_j(y)$, intermediate value theorem assures that y^* exists on the segment.

We show $\sup I < \inf J$. Suppose not. If $\sup I > \inf J$, then there exist $u_i(x) \in I$ and $u_j(y) \in J$ with $u_i(x) > u_j(y)$, a contradiction. Next, suppose $\sup I = \inf J$. Then, there exists a sequence $u_i(x^\nu) \in I$ with $u_i(x^\nu) \longrightarrow u_j(y^*)$. As we let x^ν converge to a consumption x, we have $u_i(x) = u_j(y^*)$. Taking a consumption $x' \gg x$, we have $u_i(x') > u_j(y^*)$, which contradicts the definition of y^*.

Note that each $u_j(y)$ is either $u_j(y) \leq u_i(x)$ for some x or $u_i(x) < u_j(y)$ for all $x \in R^l_+$. Thus, either $u_j(y) \leq \sup I$ or $\inf J \leq u_j(y)$ holds. As $\sup I < \inf J$, this implies that u_j is not a continuous function, a contradiction: Let us show this. Take the segment between 0 and y^*. Consider the value of $u_j(y)$, taking y on the segment. As $\sup I < \inf J$ and $y < y^*$, $u_j(y)$ takes values equal to or less than $\sup I$ before arriving at y^* and jumps to $\inf J$ suddenly when reaching y^*, which implies that u_j is not a continuous function. ■

Independence of (i)-(iii) in Lemma 4:

(ii)$\not\Rightarrow$(i): Let z be such that $z \in PO(\succcurlyeq)$ with $z_i \neq 0$ for all i, and $z \notin EIW(\succcurlyeq)$. We define a representation of $\succcurlyeq_E$ such that $u_i(0) = 0$ and $u_i(z_i) = 1$

for all i. Theorem 1 implies $z \in LME(\succsim_E)$, the desired result.

(ii)$\not\Rightarrow$(iii): Let z and p be such that $z \in PO(\succsim, p)$, $z_i \neq 0$ for all i, and $pz_1 > pz_2$. Using money-metric utility functions u_i^p, we define an extended preference such that

$$(x, i) \succsim_E (y, j) \Longleftrightarrow \alpha_i u_i^p(x) \geq \alpha_j u_j^p(y),$$

where α_i and α_j are positive constants such that $\alpha_i pz_i = \alpha_j pz_j$. Theorem 1 shows $z \in LME(\succsim_E)$. Suppose that $\succsim_E$ is a price indicator type at p and z. As Lemma 1 says $(z_1, 1) \sim_E (z_2, 2)$, the supposition implies $pz_1 = pz_2$, which is a contradiction.

(iii)$\not\Rightarrow$(i): Let $z = (\Omega, 0, ..., 0)$ and $p \in int.\Delta^l$ be such that $z \in PO(\succsim, p)$. We invoke $\succsim_E^p$ of Remark 1, a rice indicator type at p and z. Then, we have $z \notin EIW(\succsim, p)$.

(iii)$\not\Rightarrow$(ii): We use the same example as the case of (iii)$\not\Rightarrow$(i). By noting E.1 and E.2, we have $(z_1, 1) \succ_E (z_i, i)$ for all $i \neq 1$. Invoking Theorem 1, we have $z \notin LME(\succsim_E)$.

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