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Bruce McGough, Ryuichi Nakagawa

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関西大学ソシオネットワーク戦略研究機構

The Research Institute for Socionetwork Strategies,
Kansai University
Joint Usage / Research Center, MEXT, Japan
Suita, Osaka, 564-8680, Japan
URL: http://www.kansai-u.ac.jp/riss/index.html
e-mail: riss@ml.kandai.jp
tel. 06-6368-1228
fax. 06-6330-3304

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Bruce McGough
Department of Economics
University of Oregon

Ryuichi Nakagawa[2]
Faculty of Economics
Kansai University

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[2]Address: Faculty of Economics, Kansai University, 3-3-35 Yamate Suita, Osaka 564-8680, Japan. Phone: +81-6-6368-0590. Fax: +81-6-6339-7704. E-mail: ryu-naka@kansai-u.ac.jp. URL: http://www2.itc.kansai-u.ac.jp/∼ryu-naka/.

Stability of Sunspot Equilibria under Adaptive Learning with Imperfect Information

Abstract

This paper investigates whether the stability of sunspot equilibria under learning is affected by the imperfect information sets of exogenous variables held by learning agents. It finds that imperfect information under learning creates *expectational frictions* that make the adjustment of expectations easy to converge. Then, the existence of imperfect information has a positive effect on the stability of sunspot equilibria. Specifically, the stability conditions are relaxed as the degree of information imperfection (or the magnitude of the expectational frictions) increases. The paper also finds that this effect is significant in calibrated New Keynesian models. Under highly imperfect information, sunspot equilibria are stable if and only if the Taylor principle is violated, while under perfect information, implausible conditions are obtained. Our results suggest that imperfect information under learning increases the possibility of self-fulfilling fluctuations. The expectational frictions may help establish the business cycle models that explain the recent global financial fluctuations.

JEL classification: C62; D82; D83; E32; E52

Keywords: Sunspot equilibria; Imperfect information; Stability; Adaptive learning; Expectational frictions; Taylor principle

1 Introduction

Sunspot-driven business cycle models that are based on rational expectations (RE) are popular tools to account for the dynamics of business cycles that are not justified by fundamental reasons alone. The US economy in the pre-Volcker period is the most popular example of a business cycle that is explained by self-fulfilling expectations that are driven by non-fundamental reasons, or so-called "sunspots" (see Benhabib and Farmer, 1994; Farmer and Guo, 1994). These expectations are considered to have been supported by the Fed's passive stance toward inflation (Clarida, Gali, and Gertler, 2000; Lubik and Schorfheide, 2004).[1]

Models with sunspot equilibria also help investigate the recent global financial fluctuations: for example, the US housing market in the middle 2000s (Kashiwagi, 2014), the Great Recession in the late 2000s (Farmer, 2012), and the European debt crisis in the early 2010s (Bacchetta, Tille, and van Wincoop, 2012). In the theoretical literature, these phenomena are described as self-fulfilling fluctuations caused by financial frictions based on the imperfect information of fundamental variables (Benhabib and Wang, 2013; Gertler and Kiyotaki, 2015; Benhabib, Dong, and Wang, 2018)

Accordingly, the dynamics of sunspot equilibria has been investigated in alternative mechanisms of expectations formation. Sunspot equilibria might not be attainable under reasonable expectations. To test the empirical plausibility of these models, a strand of the literature investigates the stability of sunspot equilibria under *adaptive learning*, which is to form their forecasts by estimating econometric models without knowledge of the economic structure (Evans and Honkapohja, 2001). Then, in New Keynesian (NK) models, for example, the literature finds the monetary policy rules that cause stable sunspot equilibria (Carlstrom and Fuerst, 2004; Honkapohja and Mitra, 2004; Evans and McGough, 2005b, 2010; Airaudo and Zanna, 2010).

While the dynamics of sunspot equilibria have been investigated in the adaptive learning framework, how the imperfect information of fun-

[1]Non-fundamental fluctuations under the passive stance of central banks have also been suspected in the context of the EU economy during the 1980s and 1990s (Hirose, 2013) and the Chinese economy during the 1990s and 2000s (Zheng and Guo, 2013).

damental variables affects the stability of sunspot equilibria has not been fully clarified. Most of the learning literature assumes that agents have perfect information sets of economic variables, which enable estimation of the correctly specified econometric models that have the same form as RE solutions. However, this is in contrast with the above RE literature considering sunspot equilibria with imperfect information and the evidence of the imperfect information of fundamental variables.[2] Rather, in the learning framework, considering such imperfect information may be more reasonable because it is often difficult to identify the processes of fundamental variables in the absence of knowledge of the economic structure.[3] In this situation, agents' forecasting models must be underparameterized and sometimes mutually different, and misspecified expectations might affect the dynamics of sunspot equilibria. To derive implications regarding the recent financial turmoil, which might have occurred in the presence of imperfect information, the stability of sunspot equilibria should be investigated given the same information structure.

This paper investigates the stability of sunspot equilibria under adaptive learning with imperfect information. We incorporate the imperfect information sets of exogenous variables held by learning agents into a reduced-form linear expectational model and examine whether the stability conditions of sunspot equilibria are affected by the existence of imperfect information. By specifying the degree of imperfection in agents' information sets, the paper shows the analytical relationship between the degree of imperfection and the stability conditions.

Next, using a basic NK model, the paper explores how imperfect information performs in standard business cycle models. The learning literature has shown that sunspot equilibria in NK models are stable only if the central bank follows the *Taylor principle*, which is to raise the nominal interest rate actively in response to increases in economic vari-

[2]See Mankiw, Reis, and Wolfers (2003) and Madeira and Zafar (2015) for the evidence of imperfect information. Beckworth and Hendrickson (2019) find that because of imperfect information regarding output gap, its forecast error accounts for up to 13% of the fluctuations in the output gap.

[3]This difficulty is argued by Nakamura and Steinsson (2018), Giannone and Reichlin (2006), and Alessi, Barigozzi, and Capasso (2011). The fact that the consensus regarding the source of the Great Depression in the 1930s was only reached six decades later, in the 1990s, is a classic example of this type of difficulty (see Eichengreen, 1992).

ables (normally, the inflation rate).[4] However, this conventional result contradicts the aforementioned empirical evidence that during past non-fundamental fluctuations, central banks took a passive stance toward inflation. This contradiction is called the *stability puzzle* and implies the empirical *im*plausibility of NK models.[5] With this background, this paper not only demonstrates the significance of imperfect information for stability conditions imposed on monetary policy rules, but also examines whether imperfect information provides stability conditions that are consistent with the empirical evidence.

This paper finds that imperfect information under learning creates *expectational frictions* that make the adjustment of expectations sluggish so that estimated parameters converge easily. As a result, the existence of imperfect information has a *positive* effect on the stability of sunspot equilibria. That is, their stability conditions are relaxed by the existence of imperfect information and are more relaxed as the degree of information imperfection (or the magnitude of the expectational frictions) increases. Thus, even if sunspot equilibria are unstable under perfect information, they can be stable under imperfect information.

This result suggests that imperfect information under learning raises the possibility of self-fulfilling business cycle fluctuations. As is mentioned above, relationships between imperfect information and sunspot equilibria are emphasized in the recent RE literature, but most of them are mainly supported by financial frictions owing to imperfect information, which improve the stationarity of sunspot equilibria. In our model, by contrast, the relationship is supported by the expectational frictions owing to imperfect information under learning, which improve the stability of sunspot equilibria. These frictions may help characterize the dynamics of non-fundamental fluctuations under reasonable expectations.

Next, in the NK model, this paper finds that imperfect information not only relaxes stability conditions imposed on monetary policy rules, but also makes those conditions consistent with the observations during past

[4]See Honkapohja and Mitra (2004), Evans and McGough (2005b), Airaudo and Zanna (2010), and Airaudo, Nistico, and Zanna (2015).

[5]The stability puzzle was initially found in RBC models in the form that sunspot equilibria in calibrated RBC models are always unstable under adaptive learning (Branch and McGough, 2004; Evans and McGough, 2005a; Duffy and Xiao, 2007; Ji and Xiao, 2018).

business cycle fluctuations. In the absence of imperfect information, there is an implausible result that sunspot equilibria are *un*stable if the central bank is very passive in its response to economic variables. However, the implausible result is reduced in the presence of imperfect information, and if information is highly imperfect, sunspot equilibria are always stable in the violation of the Taylor principle. This result is fully consistent with past observations. Thus, imperfect information under learning can be an important mechanism to establish empirically plausible sunspot-driven business cycle models.

Our results reinforce the positive effect of imperfect information on the stability of an equilibrium under learning. Nakagawa (2015) obtains a similar result that the stability of a *fundamental* equilibrium is improved by imperfect information. In addition, the Taylor principle is the sufficient condition for the stability. Combined with these results, the present study confirms that the expectational frictions caused by imperfect information are robust both in fundamental and sunspot equilibria. Additionally, the Taylor principle ensures that the fundamental equilibrium is uniquely stable under learning.

This paper is closely related to the learning literature on the *restricted perceptions equilibrium* (RPE), in which agents' forecasting models are underparameterized and sometimes heterogeneous. The RPE is first analyzed by Marcet and Sargent (1989a), who consider the information structure of the Lucas islands model, which covers a broad class of agents' limited and mutually different information sets of economic variables. However, the stability of the equilibrium under learning was beyond the scope of their research.

While other studies focus on different information structures, Nakagawa (2015) simplifies Marcet and Sargent (1989a)'s model by limiting it to the imperfect information of exogenous variables.[6] In this structure, the degrees of limitation and heterogeneity in agents' information sets are

[6]Regarding the stability of sunspot equilibria, the literature considers the partial or asymmetric information of exogenous variables (e.g., Branch, McGough, and Zhu, 2017), the asymmetric information of endogenous variables (e.g., Adam, Evans, and Honkapohja, 2006), and the partial or asymmetric information of sunspot variables (e.g., Guse, 2005; Berardi, 2009). However, no study considers Marcet and Sargent (1989a)'s information structure, under which each agent has his/her own private information about fundamental variables as in the Lucas islands model.

specified by model parameters so that the analytical relationship between a general class of imperfect information of exogenous variables and the stability of the fundamental equilibrium is obtained.[7] However, sunspot equilibria are not analyzed. The present study fills this gap to analyze the recent financial fluctuations that are explained with a variety of imperfect information.

The paper is structured as follows. The next section presents our model and stationary sunspot equilibria attainable under rational expectations. Section 3 provides a benchmark analysis under learning with perfect information. Section 4 provides stability conditions under information imperfection to clarify the effect of imperfect information on the stability of sunspot equilibria. Section 5 applies our analytical results to a basic NK model. Finally, the paper presents our conclusions and future works.

2 Model

This section introduces a reduced-form linear expectational model. The economy is represented by the dynamics of endogenous variables $y_t = (y_{1t}, \cdots, y_{mt})'$ and serially correlated exogenous variables $w_t = (w_{1t}, \cdots, w_{nt})'$:

$$y_t = BE_t^* y_{t+1} + Cw_t, \tag{1}$$
$$w_t = \Phi w_{t-1} + v_t. \tag{2}$$

v_t is the $n \times 1$ vector of the innovations with mean zero that drive the stochastic processes of w_t. The standard deviation of w_{it} for each $i \in \{1, \cdots, n\}$ is defined by $\sigma_{ii} > 0$, and the correlation matrix of w_t is defined by $\Gamma \equiv \left(\rho_{ij}\right)_{1 \leq i,j \leq n}$, in which $\rho_{ij} \in [0, 1]$ denotes the correlation between w_i and w_j, and $\rho_{ij} = \rho_{ji}$ and $\rho_{ii} = 1$ for each $i, j \in \{1, \ldots, n\}$. The parameter B is the $m \times m$ coefficient matrix of $E_t^* y_{t+1}$, C is the $m \times n$ coefficient matrix of w_t, and Φ is the $n \times n$ matrix of the autocorrelation coefficient of w_t. E_t^* is the operator of the expectation of y_{t+1} at time

[7] Our framework is different from the *dynamic predictor selection* model (Brock and Hommes, 1997; Branch and Evans, 2006; Berardi, 2015), which assumes implicitly homogeneity in agents' information sets such that all agents choose among the same list of econometric models.

t, which is not necessarily rational. Several regularity assumptions are imposed on B, Γ, and Φ (see Appendix A): in particular, $\Gamma \geq 0$ and $\Phi \equiv diag\,(\varphi_i)_{1\leq i\leq n} \geq 0$ where $0 \leq \varphi_i < 1$ for all i.

In this model, we incorporate the *positive feedback of expectations*, that is, the eigenvalues of B have all positive real parts. Using the notation $\lambda\,[X]$ that denotes the largest value of the real parts of the eigenvalues of a matrix X, the positive feedback in this model is represented by

$$\lambda\,[-B] < 0. \tag{3}$$

The positive feedback is a typical feature of calibrated business cycle models (see NK models in Section 5.3, for example). By imposing the restriction (3), we obtain the stability conditions that are satisfied in calibrated models with the positive feedback.[8]

Under rational expectations ($E_t^* = E_t$), if and only if there exist the eigenvalues of matrix B outside the unit circle, the system (1)–(2) has the non-explosive solution of the general form,

$$y_t = B^{-1}y_{t-1} - B^{-1}Cw_{t-1} + \epsilon_t, \tag{4}$$

for agents' forecast errors $\epsilon_{t+1} \equiv y_{t+1} - E_t y_{t+1}$ which is an $m \times 1$ martingale difference sequence vector and satisfies $E_t\epsilon_{t+1} = 0$. ϵ_{t+1} is called a *sunspot* and the solution (4) is called a *stationary sunspot rational expectations equilibrium*. In this case, the equilibrium is indeterminate given an initial state of w_t.

The general form (4) can be transformed to a *common factor* (CF) representation if and only if either of the non-explosive eigenvalues is real (Evans and McGough, 2005c):

$$y_t = \bar{c}w_t + \bar{d}\xi_t, \tag{5}$$

where

$$\xi_t = \theta^{-1}\xi_{t-1} + \varepsilon_t, \tag{6}$$

$$\bar{c} = B\bar{c}\Phi + C, \quad \bar{d} = s\hat{d}. \tag{7}$$

[8]The theoretical literature derives stability conditions that need the *negative* feedback of expectations, which contradicts the positive feedback in calibrated models (Evans and Honkapohja, 2003a,b; Evans and McGough, 2005c; Shea, 2013, 2016; Berardi, 2015). This contradiction is the reason for the stability puzzle in calibrated models (see Evans and McGough, 2005a; Duffy and Xiao, 2007).

ξ_t and ε_t are the martingale difference sequences originating from sunspot ϵ_t. θ is the real eigenvalue satisfying $-1 < \theta^{-1} < 1$ such that y_t and ξ_t are stationary. $\bar{d}$ is the product of an arbitrary real constant s and the real eigenvector $\hat{d}$ corresponding to the eigenvalue θ.[9] $E(w_t \varepsilon_t) = E(w_t \xi_t) = 0$ are assumed.

A necessary condition for the existence of this form of stationary sunspot REEs is described as follows:

Lemma 1 *In the model (1)–(2), there exist stationary sunspot rational expectations equilibria of CF representation (5) only if*

$$\lambda[B] > 1. \tag{8}$$

That is, either of the eigenvalues of matrix B needs to have the real part greater than one.

Our analysis will focus on the stability of the equilibria of CF representation (5) instead of the general form (4). This is because the instability of the general form is found by Evans and McGough (2005c), and the robustness of this result to the existence of imperfect information is confirmed in our supplementary analysis. On the other hand, the stability of the form (5) is found by Evans and McGough (2005c), while its instability in models with the positive feedback of expectations is implied by Duffy and Xiao (2007). Other mixed results are also provided in different business cycle models.[10]

3 Sunspot equilibria with perfect information

This section shows a benchmark analysis on the stability of sunspot equilibria with perfect information under learning. Readers who are familiar

[9] If there are two or more real eigenvalues outside the unit circle, there also exist the other form of the CF representations (see Evans and McGough, 2005c).

[10] See Hirose (2008), Shea (2013), McGough, Meng, and Xue (2013), Airaudo (2013), Arifovic, Bullard, and Kostyshyna (2013), and Berardi (2015).

with the benchmark analysis may skip this section and directly go to Section 4.

3.1 Adaptive learning

Suppose that agents do not have enough knowledge of the economic structure to coordinate on rational expectations. An alternative mechanism to form their forecasts $E_t^* y_{t+1}$ is by estimating econometric models with available data up to time t, $\{y_s', w_s', \xi_s\}_{s=1}^t$.[11] Following the methodology of the learning literature, agents are assumed to know the functional form of sunspot equilibria (5) and estimate the perceived law of motion (PLM) of the same form:

$$y_t = cw_t + d\xi_t + e_t, \tag{9}$$

where c is the $m \times n$ matrix of coefficients for w_t, d is the m-vector of coefficients for sunspot ξ_t, and e_t is the m-vector of error terms that are perceived to be white noise.

Using the estimated parameters $\phi' \equiv (c, d)$, agents form the forecast:

$$E_t^* y_{t+1} = c\Phi w_t + \theta^{-1} d\xi_t. \tag{10}$$

The actual law of motion (ALM) of the economy is obtained by incorporating Eq. (10) into Eq. (1) as

$$y_t = (Bc\Phi + C)\, w_t + d\xi_t. \tag{11}$$

Note that $\theta^{-1}Bd = d$.

3.2 Dynamics of Parameters

In real-time learning, the parameters $\phi_t' = (c_t, d_t)$ estimated at time t are the optimal linear projections of y_{t-1} on $z_{t-1}' \equiv \left(w_{t-1}', \xi_{t-1}\right)$ that satisfy the following least-squares orthogonality condition:

$$Ez_{t-1}\left(y_{t-1} - \phi_t' z_{t-1}\right) = 0.$$

[11] Our analytical results are independent of whether contemporaneous endogenous variables y_t are used to form the forecast $E_t^* y_{t+1}$, because the forecast (10) is not determined by y_t.

Then, the local dynamics of ϕ' are governed by the associated ordinary differential equation (ODE) (see Evans and Honkapohja, 2001, chapter 6):

$$\frac{d\phi}{d\tau} = T(\phi) - \phi, \tag{12}$$

where τ denotes notional time and $T(\phi)$ is the mapping from the PLM to the ALM:

$$T(\phi) \equiv \begin{pmatrix} T_c(c) & T_d(d) \end{pmatrix} = \begin{pmatrix} Bc\Phi + C & d \end{pmatrix}.$$

If the ODE is locally asymptotically stable around the fixed point $\bar{\phi}$, the parameters ϕ_t converge to the fixed point (7) under real-time learning, and the economy is stable around a sunspot equilibrium (5). $\bar{d}$ is arbitrary so that there exists a continuum of sunspot equilibria.

3.3 Stability condition

The ODE (12) is locally stable if and only if the eigenvalues of the Jacobians regarding c,

$$D(T_c(c) - c) = \Phi \otimes B - I_{mn},$$

have negative real parts:

$$\lambda[B] < \lambda[\Phi]^{-1}. \tag{13}$$

Combined with the stationary condition (8), the stability condition of stationary sunspot equilibria is provided as follows:

Proposition 1 *In the system (1)–(2) with perfect information, stationary sunspot equilibria (7) are locally stable under learning only if*

$$1 < \lambda[B] < \lambda[\Phi]^{-1}. \tag{14}$$

Note that $\lambda[\Phi] = \max\{\varphi_i\}_{1\leq i\leq n} < 1$.

The proposition implies that sunspot equilibria in calibrated business cycle models tend to be unstable under perfect information. The parameter region (14) does not seem wide enough to include the values of calibrated parameters. If the largest autocorrelation of fundamental shocks (that is, $\lambda[\Phi]$) is significantly large such as 0.9, then Eq. (14) is $1 < \lambda[B] < 1.11\cdots$. This leads to most of the instability results (that is, the stability puzzle) that have been observed in calibrated business cycle models with perfect information (see Section 5.3).[12]

4 Sunspot equilibria with imperfect information

In what follows, the assumption of perfect information is relaxed by introducing the imperfect information of exogenous variables.

4.1 Imperfect information

To analyze a variety of the information structures of exogenous variables, our model incorporates the following private information, which is considered in Nakagawa (2015):

Assumption 1 *For each $i \in \{1, ..., n\}$, the evolution of the exogenous variable $\{w_{is}\}_{s=1}^{t}$ is observable for agents of type i and unobservable for agents of other types.*

That is, each exogenous variable of the economy is privately observable for a part of agents. Implicitly, it is assumed that different types cannot share any information of unobservable variables. Then, agents of type i recognize the stochastic characteristics of observable variable w_{it}, but do not recognize the characteristics of the unobservable variables $\{w_{jt}\}_{j=1,j\neq i}^{n}$.[13]

[12] The other reason is that most studies include constant terms in PLMs so that sunspot equilibria are always unstable in models with the positive feedback of expectations. Evans and McGough (2005a) and Duffy and Xiao (2007) show the corresponding instability results in RBC models.

[13] In particular, the stochastic distributions of $\{w_j\}_{j\neq i}^{n}$, the correlations of exogenous variables $\{\rho_{ij}\}_{i,j=1,i\neq j}^{n}$, and the number n of exogenous variables are unobservable for other types of agents. If their information were common knowledge, agents could use it in adaptive learning, which should be different from the form described in this paper.

For simplicity, the population of each type is assumed to be the same at $\frac{1}{n}$.

This structure is originally considered in the Lucas island model and analyzed under adaptive learning by Marcet and Sargent (1989a). This type of private information describes a feature of the information of individual fundamental shocks: for example, a preference shock possessed by a household (see Allen and Gale, 2004) and the profitability of a borrower in a financial market (see Stiglitz and Weiss, 1981).

The advantage of this structure in the analysis of learning is to cover not only private information, but also a variety of the information structures of exogenous variables. For example, if $n = 1$ (no limitation) or $\rho_{ij} = 1$ (and hence $\varphi_i = \varphi_j$ and $\frac{w_{it}}{\sigma_{ii}} = \frac{w_{jt}}{\sigma_{jj}}$) for all i, j (no heterogeneity), the information sets of all types are essentially reduced to the perfect ones in Section 3 so that the analysis of this section covers the benchmark analysis of Section 3 as a special case. In addition, a broad class of imperfect information of exogenous variables considered in the literature is reproduced by accommodating the characteristics of $\{w_{it}\}_{i=1}^{n}$ (here, n, $\left\{\rho_{ij}\right\}_{i,j=1}^{n}$, and $\{\varphi_i, \sigma_{ii}\}_{i=1}^{n}$).[14] The number n of exogenous variables may define the degree of *limitation* in the information set of each agent; the larger n is, the more limited each information set is relative to the full one.[15] The value $1 - \rho_{ij}$ (or the correlation ρ_{ij} of the two exogenous variables $\{w_{it}, w_{jt}\}$ for each $i, j \in \{1, ..., n\}$) may define the degree of the *heterogeneity* (or *homogeneity*) in the information sets of types i and j; the greater $1 - \rho_{ij}$ is, the more heterogeneous both information sets are.[16] Later, both degrees will be combined together to define *the degree of information imperfection* with a single scalar measure.

[14]Nakagawa (2015) shows that this framework covers, for example, partial information sets seen in the RPE literature, asymmetric information sets, and different populations of the types of agents.

[15]For private information to exist for each type of agent, the number of unobservable variables $(n - 1)$ must be greater than the number of endogenous variables (m). For example, if $m = 2$, $n > 3$ must hold.

[16]Although the degree of heterogeneity in the information sets of all types cannot be represented by a single scalar measure, we will say "the degree of heterogeneity in the information sets of all types increases" if ρ_{ij} falls for at least one (i, j) and falls or remains the same for every (i, j).

4.2 Heterogeneously misspecified learning

Let us describe adaptive learning by the agent of type i with the imperfect information set $\{y'_s, w_{is}, \xi_s\}_{s=1}^{t}$, which is limited and different from the information sets held by other types in terms of $\{w_{it}\}_{i=1}^{n}$. Let us assume that the agent recognizes the correct form of sunspot equilibria (5), but that the agent is constrained to specify an underparameterized PLM:[17]

$$y_t = c_i w_{it} + d_i \xi_t + e_{it}. \tag{15}$$

For simplicity, we assume that agents of all types believe the same sunspot ξ_t. Using the estimated parameters, the forecast of agent i at time t is formed as

$$E^*_{it} y_{t+1} = c_i \varphi_i w_{it} + \theta^{-1} d_i \xi_t, \tag{16}$$

where E^*_{it} is the operator of forecasts formed by type i at time t.

The PLM and the forecast of each type are *heterogeneously misspecified* to the same degree as the degrees of information limitation and heterogeneity. The *degree of misspecification* in the PLM/forecast of each type is specified one-to-one by the degree of limitation of each information set (that is, n). The *degree of heterogeneity* in the PLMs/forecasts of types i and j is specified one-to-one by the degree of heterogeneity of information sets of both types (that is, $1 - \rho_{ij}$).

The forecast $E^*_t y_{t+1}$ in Eq. (1) is determined by the average of the forecasts of all types of the form (16). Following the same populations of different types, let us assume that forecasts of different types $\{E^*_{it} y_{t+1}\}_{i=1}^{n}$ have equal contributions to the dynamics of the economy. Then, the forecast $E^*_t y_{t+1}$ is formulated as follows:

$$E^*_t y_{t+1} = c\Phi w_t + \theta^{-1} d \xi_t, \tag{17}$$

where E^*_t is the operator of the average of heterogeneous forecasts $E^*_t = \frac{1}{n}\sum_{i=1}^{n} E^*_{it}$. $d \equiv \frac{1}{n}\sum_{i=1}^{n} d_i$ is the average of the coefficient vectors for all types, and $c \equiv \frac{1}{n}(c_1, \cdots, c_n)$ is the $m \times n$ matrix that combines the

[17] Note that in the presence of imperfect information, it is reasonable to include y_{t-1} in the PLM (15) because uninformed agents extract some information about unobservable variables from the process of y_t. Our supplementary analysis finds that, in our purely forward-looking model, the effect of imperfect information on the stability is robust to the inclusion of y_{t-1} in the PLM (that is, the VAR(1) PLM).

coefficient vectors $\{c_i\}_{i=1}^n$ of the PLMs of the form (15) for all types and multiplies them by the proportion $\frac{1}{n}$.

The ALM is obtained by substituting Eq. (16) into the system (1)–(2):

$$y_t = (Bc\Phi + C)\, w_t + d\xi_t. \tag{18}$$

In this framework, there emerge *frictions* in the expectations formation. Imperfect information under learning causes the same degrees of misspecification and heterogeneity in the PLMs of individual types. As a result, given $\{c_i\}_{i=1}^n$, the aggregate forecast (17) is less responsive to the evolutions of $\{w_{it}\}_{i=1}^n$ than those under perfect information. In response, the adjustment of the economy (i.e., the ALM (18)) is also reduced.

4.3 Dynamics of Parameters

As in Section 3, the parameters $\phi_{it}' = (c_{it}, d_{it})$ of type i are the optimal linear projections of y_{t-1} on $z_{i,t-1}' \equiv \left(w_{i,t-1}, \xi_{i,t-1}\right)$ that satisfy the following least-squares orthogonality condition,

$$Ez_{i,t-1}\left(y_{t-1} - \phi_{it}' z_{i,t-1}\right) = 0,$$

such that the misspecification in the PLM (15) is not detected.

The local dynamics of ϕ_i in real-time learning is inferred from the stochastic recursive algorithms of ϕ_i formulated by the PLM (15) and the ALM (18). As a result, the dynamics of the parameters of all types are represented by the dynamics of the aggregate parameters $\phi' = (c, d)$, which are governed by the following ODE:

$$\frac{d\phi}{d\tau} = T(\phi) - \phi, \tag{19}$$

where

$$\begin{aligned} T(\phi) &\equiv \begin{pmatrix} T_c(c) & T_d(d) \end{pmatrix} \\ &= \begin{pmatrix} (Bc\Phi + C)\left(\frac{1}{n}\Psi\right) & d \end{pmatrix}, \\ \Psi &\equiv diag\,(\sigma_{ii})_{1\le i\le n} \cdot \Gamma \cdot diag\,(\sigma_{ii})_{1\le i\le n}^{-1}. \end{aligned}$$

The derivation of the ODE is shown in Appendix B. Mapping $T(\phi)$ provides the coefficients of the forecasts of y_t updated by agents of all types.

The fixed points $\bar{\phi}' = \left(\bar{c}, \bar{d}\right)$ are

$$\bar{c} = \left(B\bar{c}\Phi + C\right)\left(\frac{1}{n}\Psi\right), \quad \bar{d} = s\hat{d}. \tag{20}$$

If the ODE is locally stable around the fixed point $\bar{\phi}$, the aggregate parameters ϕ_t converge to the fixed point under real-time learning, and the economy is determined at the equilibrium with Eq. (20). The fixed point $\bar{d}$ is arbitrary so that the sunspot shock ξ_t drives the economy arbitrarily.

Therefore, the sunspot equilibrium in the existence of imperfect information is defined as a stochastic process for $\{y_t\}_{t=0}^{\infty}$ following the system (1)–(2) given that $\{E_t^* y_{t+1}\}_{t=0}^{\infty}$ is the average of the forecasts formed by the PLMs of the form (15) for all i with the parameters $\{\phi_i' = (c_i, d_i)\}_{i=1}^{n}$ determined at the fixed point (20) of the ODE (19).

4.4 Stability condition

The stability of sunspot equilibria is subject to whether the aggregate parameters $\phi' \equiv (c, d)$ converge to the fixed point. They are locally stable if and only if the eigenvalues of the Jacobian of the associated ODE (19) regarding c,

$$D\left(T_c\left(c\right) - c\right) = \left(\Phi\left(\frac{1}{n}\Psi\right)\right)' \otimes B - I_{mn},$$

have all negative real parts:

$$\lambda\left[B\right] < \mu\lambda\left[\Phi\right]^{-1}, \tag{21}$$

where

$$\mu \equiv \frac{\lambda\left[\Phi\right]}{\lambda\left[\Phi\left(\frac{1}{n}\Psi\right)\right]} = \frac{\lambda\left[\Phi\right]}{\lambda\left[\Phi\left(\frac{1}{n}\Gamma\right)\right]}.$$

Thus, combined with the stationary condition (8), the stability condition under imperfect information is given as follows:

Proposition 2 *In the system (1)–(2) with imperfect information, stationary sunspot equilibria (20) are locally stable under learning only if*

$$1 < \lambda\left[B\right] < \mu\lambda\left[\Phi\right]^{-1}. \tag{22}$$

Comparing with the stability condition under perfect information (14), this result clarifies that the effect of imperfect information is represented by the parameter μ. In our mechanism, imperfect information under learning creates the *expectational frictions* that make the forecast and the ALMs less responsive to the evolutions of exogenous variables. Thus, the parameter μ represents the magnitude of the frictions caused by imperfect information.

4.5 Effect of imperfect information

Here, let us find the relationship between imperfect information and the stability of sunspot equilibria.

The parameter μ includes all the parameters of the degrees of information limitation and heterogeneity (n and $\left\{1-\rho_{ij}\right\}_{i,j=1}^{n}$). Those parameters have the following relationships (see Nakagawa, 2015, Lemma 1 & Remark 1):

Lemma 2 *For each $n \geq 1$ and $i, j \in \{1, \cdots, n\}$,*

1. *$\mu \geq 1$ with equality iff $n = 1$ or $\rho_{ij} = 1$ for all i, j;*
2. *$\frac{d\mu}{d\left(1-\rho_{ij}\right)} \geq 0$ for all i, j;*
3. *$\frac{d\mu}{dn} \geq 0$ if $\varphi_i = \varphi \in [0, 1)$ and $\rho_{ij} = \rho \in [0, 1]$ for all i, j and $i \neq j$.*

Lemma 2.1 means that $\mu = 1$ under perfect information ($n = 1$ or $\rho_{ij} = 1$ for all i, j) and $\mu > 1$ under imperfect information. Lemmas 2.2 & Lemma 2.3 mean that the parameter μ has nonnegative monotonic relationships with $\left\{1-\rho_{ij}\right\}_{i,j=1}^{n}$ and, in the case of similar exogenous variables, n. Thus, the parameter μ defines *the degree of information imperfection* of the whole economy, which increases the magnitude of the expectational frictions.

Thus, the existence of imperfect information under learning has a *positive* effect on the stability of sunspot equilibria as follows:

Proposition 3 *The stability condition (22) is relaxed upwards by the existence of imperfect information.*

More specifically, this effect is raised by the degree of information imperfection as follows:

Proposition 4 *For each $n \geq 1$ and $i, j \in \{1, \cdots, n\}$, the stability condition (22) is relaxed upwards by an increase in the degree of information imperfection (or the magnitude of the expectational frictions) μ, which is increased or unchanged by an increase in the degree of heterogeneity $1 - \rho_{ij}$ in the information sets of types i and j for each i, j or by an increase in the degree of limitation n of the information set of each type if exogenous variables have the same stochastic characteristics as in Lemma 2.3.*

That is, the degrees of information heterogeneity and limitation have nonnegative effects on the stability.

The mechanism of imperfect information under learning that creates the expectational frictions is intuitively apparent. Under imperfect information, the aggregate forecast and the economy are less responsive to the evolutions of exogenous variables than under perfect information. At the end of the period, the sluggish adjustment of the economy reduces the updating of the parameters in PLMs. In the next period, this reduction in the updating gives a feedback to reduce the adjustment of the economy. The interaction between the adjustment of the economy and the updating of the parameters makes the estimated parameters converge easily so that the stability of sunspot equilibria is improved.

The expectational frictions are illustrated in brief calibrations. Figure 1 shows the calibrations of the updating of the parameter c_{it} and the adjustment process of endogenous variable y_t in response to a one-time evolution of $w_{1,0} = 1$ under different degrees of information heterogeneity. Here, with the degree of limitation fixed as $n = 10$, the degree of heterogeneity $1 - \rho$ represents the degree of information imperfection of the economy and the magnitude of the expectational frictions. We find that the response of y_t to the evolution of w_1 are reduced as the degree of heterogeneity increases. This reduction makes the updating of c_t sluggish

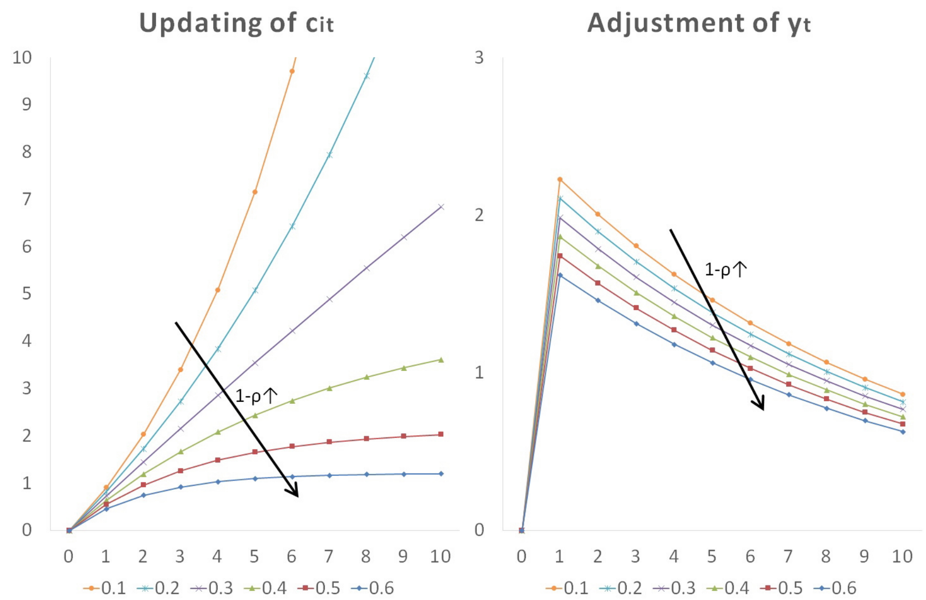

Figure 1: The updating of the parameter c_{it} and the adjustment processes of y_t under different degrees of information heterogeneity. Consider the univariate case ($m = 1$) and assume $B = 1.5$, $C_i = 1$, $E\left(w_{it}^2\right) = 1$ for all i, $E\left(\xi_t^2\right) = 1$, and the same stochastic characteristics of exogenous variables as in Lemma 2.3: $\varphi_i = 0.9$ for all i and $\rho_{ij} = \rho$ for all $i \neq j$. The initial values of parameters (c, d) are set at $(0, 1)$. The degree of limitation (n) is fixed at 10. Then, the stability condition (22) is $1 - \rho > 0.29$.

so as to converge. On the other hand, if $1 - \rho$ is lower than the threshold of the stability condition, the response of y_t and the updating of c_t are so large as to explode. In this way, the expectational frictions affect the stability of sunspot equilibria.

Note that the expectational frictions have no effect on the stability if *all* agents include constant terms in their PLMs (e.g., $y_t = a_i + c_i w_{it} + d_i \xi_t + e_{it}$).[18] This case can occur if all agents have no knowledge of the steady state of y_t, which corresponds to the fixed points of $\{a_i\}_{i=1}^n$. However, the observable steady state is a standard assumption in business cycle models.[19] Further, the frictions hold the effect if there are at least

[18] In this case, $\lambda[B] < 1$ is added as the convergence condition of a_i, which is more restrictive than Eq. (21) enough to eliminate the effect of the expectational frictions.

[19] In NK models, for example, the steady state of output is assumed to be known by agents so that the central bank can control the nominal interest rate in response to

a small proportion of agents who know the steady state.[20] Thus, our results are likely to be robust in reasonable situations.

4.6 Implications

These results provide several implications. First, our results suggest that the imperfect information of fundamental shocks raises the possibility of self-fulfilling business cycle fluctuations. Relationships between them have been shown in a bunch of the RE literature, where financial frictions due to imperfect information, for example, improve the *stationarity* of self-fulfilling expectations (Benhabib and Wang, 2013; Gertler and Kiyotaki, 2015; Benhabib, Dong, and Wang, 2018). Our results, on the other hand, show that the expectational frictions based on imperfect information under learning improve the *stability* of those expectations.

Secondly, our results imply that the time-variation in the information structure can drive boom-bust cycles. Proposition 4 shows the positive relationship between the degree of imperfection in agents' information sets and the stability of sunspot equilibria. That is, self-fulfilling fluctuations may occur when agents' information sets become imperfect and must end when information is recovered close to perfect. Such time-variations are observed in structural changes of the economy. Recent technological (or financial) innovations, for example, were initially unfamiliar to people (or familiar to only a part of people) and to become familiar gradually.[21]

an output gap, which is the difference between actual output and the steady state of output. The steady state of the inflation rate is also determined by the central bank as the inflation target and reported publicly.

[20] If all agents include constant terms in their PLMs, the convergence condition of a_i is $\lambda\,[B] < 1$ (see footnote 18). On the other hand, if the proportion of the agents who know the steady state and do not include constant terms is $p \in [0, 1]$, the above convergence condition is modified to $\lambda\,[B] < (1-p)^{-1}$ while the condition (21) of c is unchanged. Notice that if $p > 1 - \lambda\,[\Phi]$, the convergence conditions are represented by Eq. (21) which is affected by imperfect information. When, for example, $\varphi_i = 0.9$ for all i, this case occurs if $p > 1 - \lambda\,[\Phi] = 0.1$ (that is, at least ten percent of agents know the steady state).

[21] The synchronizations of technological or financial innovations and changes in the information structure are observed before past non-fundamental fluctuations. For example, Greenwood and Nagel (2009) find that *inexperienced* investors play a role in the formation of the US IT bubbles in the late 1990s. Mizen (2008) argues that before the US subprime bubble in the 2000s, market investors *did not fully understand* the risk inherent in new financial products and mortgage lenders also *did not have*

Our results suggest that those informational structural changes might have caused boom-bust cycles by generating time-variations in information structures.

Thirdly, in such a time-variation in the information structure, a given government policy could fail to prevent self-fulfilling fluctuations under high imperfection even if the policy succeeded in preventing them under perfect information. Under high imperfection, the government should adopt a restrictive policy that increases $\lambda[B]$, that is, the size of the expectational feedback.

Finally, our results also propose the mechanism of the expectational frictions caused by imperfect information as an important devise to establish plausible sunspot-driven business cycle models. In the literature, there is the stability puzzle that sunspot equilibria in calibrated business cycle models are always unstable under learning. This implies the empirical *im*plausibility of those models. However, Proposition 4 implies that in the presence of highly imperfect information, the stability condition of sunspot equilibria might be satisfied in the ranges of calibrated parameters. Thus, incorporating the expectational frictions may describe non-fundamental fluctuations in plausible frameworks of expectations.

5 Application to a NK model

This section shows how the expectational frictions caused by imperfect information perform in a NK model. Further, let us discover whether they provide the results that are consistent with the historical evidence.

5.1 NK model

The NK model is

$$x_t = -\alpha\left(i_t - E_t^* \pi_{t+1}\right) + E_t^* x_{t+1}, \tag{23}$$

$$\pi_t = \kappa x_t + \beta E_t^* \pi_{t+1}. \tag{24}$$

sufficient information about rapidly increasing subprime borrowers. Nakagawa and Uchida (2011) suggest that the Japanese financial deregulation since the early 1980s contributed to the asset-price bubble in the late 1980s by inducing banks to follow the lending behavior of banks that were *more informed about new borrowers* in the early to mid 1980s.

The model has three endogenous variables: output gap x_t, the inflation rate π_t, and the nominal interest rate i_t. Eq. (23) is a log-linearized intertemporal Euler equation that is derived from the households' optimal choice of consumption. Eq. (24) is a Phillips curve with the forward-looking component that is derived from the optimizing behavior of monopolistically competitive firms with Calvo price setting. $\alpha > 0$, $\kappa > 0$, and $0 < \beta < 1$ are assumed.

For robustness, let us consider three types of nominal interest rate rules that are popular in the literature: a *contemporaneous* nominal interest rate rule (see Lubik and Schorfheide, 2004),

$$i_t = \phi_\pi \pi_t + \phi_x x_t + w_t, \tag{25}$$

a *forward*-looking rule (McCallum and Nelson, 1999),

$$i_t = \phi_\pi E_t^* \pi_{t+1} + \phi_x E_t^* x_{t+1} + w_t, \tag{26}$$

and a *semi-forward*-looking rule (Clarida, Gali, and Gertler, 2000),

$$i_t = \phi_\pi E_t^* \pi_{t+1} + \phi_x x_t + w_t, \tag{27}$$

where w_t is the monetary policy shock. The parameters ϕ_π and ϕ_x are controlled by the central bank, and $\phi_\pi, \phi_x \geq 0$ are assumed.

To introduce the imperfect information sets of exogenous variables, the policy shock w_t is assumed to be the aggregation of individual monetary policy shocks: $w_t \equiv \sum_{i=1}^{n} w_{it}$. These shocks describe, for example, the preference shocks of different policy board members of the central bank.[22] The shock w_{it} for each $i \in \{1, \cdots, n\}$ follows a persistent process: $w_{it} = \varphi_i w_{i,t-1} + v_{it}$, where $0 \leq \varphi_i < 1$ and the disturbance term v_{it} has a zero mean. The correlation of w_{it} and w_{jt} is $\rho_{ij} \geq 0$ for each $i, j \in \{1, \cdots, n\}$. If w_{it} for each i is observable for all agents, they specify correctly specified PLMs and homogeneous forecasts as in Eqs. (9)–(10). If w_{it} for each i is privately observable for $\frac{1}{n}$ of agents and unobservable for other agents, they specify underparameterized PLMs and heterogeneous forecasts as in Eqs. (15)–(16). Following the assumptions given by Branch

[22] In this example, each policy rule may be interpreted as the average of the policy reaction functions of different members, and w_t in each equation as the average of exogenous beliefs of different members (see Riboni and Ruge-Murcia, 2008).

and McGough (2009), the aggregate forecasts $(E_t^* x_{t+1}, E_t^* \pi_{t+1})$ are given by the averages of the forecasts of all types $\{(E_{it}^* x_{t+1}, E_{it}^* \pi_{t+1})\}_{i=1}^n$ as in Eq. (17).[23]

In this model, the stationary condition (Lemma 1) is described as follows:

Lemma 3 *In the NK model (23)–(24) with any of the nominal interest rate rules (25)–(27), there exist stationary sunspot equilibria if and only if*

$$\kappa (\phi_\pi - 1) + \phi_x (1 - \beta) < 0. \tag{28}$$

The proof is shown in Appendices C, D, and E. Note that the stationary condition (28) is not only the necessary, but also the sufficient condition because Eq. (28) ensures that the eigenvalues of the coefficient matrix are all real. Notice also that the lemma holds under any of the policy rules. Eq. (28) corresponds to the violation of the so-called *Taylor principle* that prevents stationary sunspot equilibria under rational expectations.[24]

5.2 Stability conditions

Let us obtain stability conditions imposed on the monetary policy rules. First, consider the contemporaneous rule (25). The stability condition is provided by Proposition 2:

Proposition 5 *In the NK model (23)–(24) with the contemporaneous rule (25), stationary sunspot equilibria are locally stable under learning if and only if*

$$-\frac{1}{\alpha}\left(1 - \mu^{-1}\lambda[\Phi]\right)\left(1 + \alpha\kappa + \alpha\beta\phi_x - \beta\mu^{-1}\lambda[\Phi]\right) < \kappa (\phi_\pi - 1) + \phi_x (1 - \beta) < 0. \tag{29}$$

[23] Individual monetary policy shocks $\{w_{it}\}_{i=1}^n$ keep the decision rules of agents underlying the NK model identical. Then, the NK model holds the original form while the aggregate forecasts are replaced with the average of the forecasts of different agents (see Branch and McGough, 2009).

[24] Without the positive feedback restriction (3), the violation of the Taylor principle (28) is a sufficient, but not necessary condition for indeterminacy under the forward- and semi-forward-looking rule. See Bullard and Mitra (2002, Propositions 1 & 4) and Woodford (2003, Propositions 4.3 & 4.5) in detail.

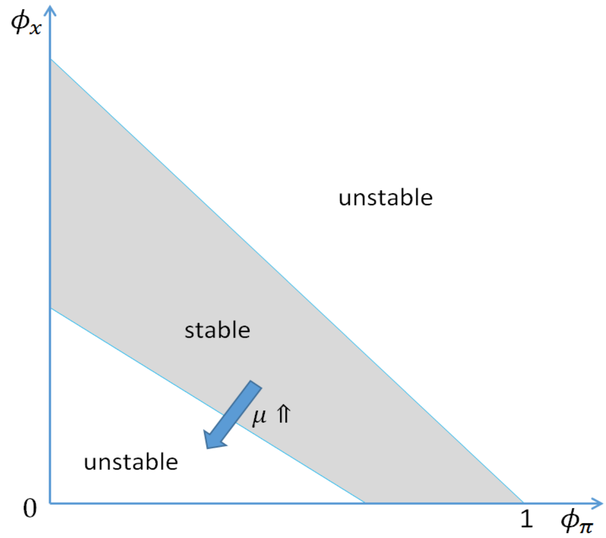

Figure 2: Stability condition under the contemporaneous rule.

The proof is shown in Appendix C. The left-hand-side comes from the convergence condition.

The stability condition (29) is described in Figure 2. We find that sunspot equilibria are stable only if the policy rule is passive such that it violates the Taylor principle. This feature is consistent with the historical evidence that during the past business cycles that were suspected to be non-fundamental, central banks took the passive stance toward economic variables.

The figure demonstrates that the expectational frictions caused by imperfect information expand the region of stability downwards. According to Proposition 4, the lower bound of the stability condition (29) receives the downward effect that increases with the magnitude of the frictions μ. In this sense, those frictions raise the possibility of sunspot equilibria in the NK model as well.

In addition, the expectational frictions improve the plausibility of NK models with sunspot equilibria. In the absence of information imperfection, the parameter region of stability has the implausible part where sunspot equilibria are unstable if the rule is *highly* passive. This part is reduced with the magnitude of the frictions. If information is highly imperfect, the implausible part is perfectly eliminated so that the violation of the Taylor principle is not only the necessary, but also sufficient condition for the stability. This result is further consistent with the historical evidence. Thus, those frictions can be an important devise to characterize the dynamics of past non-fundamental fluctuations.

Finally, these results also provide the policy implication that the Taylor principle is important to prevent self-fulfilling fluctuations in the presence of imperfect information. Without imperfection, the principle is not a necessary condition for preventing stable sunspot equilibria. However, the principle becomes more important as the degree of imperfection increases, and if agents' information sets are highly imperfect, the principle is exactly the necessary (and sufficient) condition for preventing them. Nakagawa (2015) shows in the same framework that the Taylor principle is the sufficient condition for the stability of the fundamental equilibrium. Thus, the present result implies that if the central bank follows the Taylor principle, the fundamental equilibrium is the unique stable equilibrium under adaptive learning.

These results are robust to the type of a policy rule. Let us also provide stability conditions under the forward-looking rule (26) and the semi-forward-looking rule (27).

Proposition 6 *In the NK model (23)–(24) with the forward-looking rule (26), stationary sunspot equilibria are locally stable under learning if and only if*

$$-\frac{\left(1-\mu^{-1}\lambda\left[\Phi\right]\right)\left(1-\beta\left(1-\alpha\phi_x\right)\mu^{-1}\lambda\left[\Phi\right]\right)}{\alpha\mu^{-1}\lambda\left[\Phi\right]} < \kappa\left(\phi_\pi-1\right)+\phi_x\left(1-\beta\right) < 0. \tag{30}$$

The proof is shown in Appendix D.

Proposition 7 *In the NK model (23)–(24) with the semi-forward-looking rule (27), stationary sunspot equilibria are locally stable under learning if* *and only if*

$$-\frac{\left(1-\mu^{-1}\lambda\left[\Phi\right]\right)\left(1+\alpha\phi_x-\beta\mu^{-1}\lambda\left[\Phi\right]\right)}{\alpha\mu^{-1}\lambda\left[\Phi\right]}<\kappa\left(\phi_\pi-1\right)+\phi_x\left(1-\beta\right)<0. \tag{31}$$

The proof is shown in Appendix E.

These conditions have the same features as the one under the contemporaneous rule (Propositions 5). Then, independent of the type of a policy rule, the expectational frictions caused by imperfect information raise the possibility of sunspot equilibria in the NK model and improve its empirical plausibility, and under the Taylor principle, the fundamental equilibrium is the unique stable equilibrium.

These results are also consistent with the empirical evidence that non-fundamental fluctuations were observed under passive contemporaneous and semi-forward-looking rules (Clarida, Gali, and Gertler, 2000; Lubik and Schorfheide, 2004), while most of the learning literature shows stable sunspot equilibria only under active forward-looking rules (Honkapohja and Mitra, 2004; Evans and McGough, 2005b; Airaudo and Zanna, 2010; Airaudo, Nistico, and Zanna, 2015).[25]

5.3 Calibrations

Finally, let us calibrate stability conditions in order to examine the significance of the expectational frictions in calibrated NK models.

Calibrated parameters (α, κ, β) in Table 1 are utilized. The number of monetary policy shocks $\{w_{it}\}_{i=1}^n$ are set as $n = 10$.[26] For simplicity, let us assume the same stochastic characteristics of the shocks as in Lemma 2.3. The autocorrelation φ is set as $\varphi = 0.9$.[27] Under these settings, let

[25] Nakagawa (2019) shows that sunspot equilibria under active forward-looking rules exhibit *oscillatory* adjustment processes, which are emphasized by Duffy and Xiao (2007) to be inconsistent with the empirical evidence of business cycles that exhibited *monotonic* or *cyclic* processes.

[26] This number refers to the numbers of the members of the US Federal Open Market Committee (12 members), the ECB Executive Board (6), and the BOJ Policy Board (9).

[27] The serial correlation of a monetary policy shock is estimated to be 0.92 by Rudebusch (2002) using US data during 1987-1999. Similar results are obtained by Consolo and Favero (2009) using the data of the pre-Volcker period. Carrillo, Feve, and Math-

Table 1: Structural & policy parameters

	Structural			Policy		
	α	κ	β	Type	ϕ_π	ϕ_x
Lubik and Schorfheide (2004)	1/1.45	0.77	0.99	c	0.77	0.17
McCallum and Nelson (1999)	0.164	0.3	0.99	f	0.87	0.17
Clarida, Gali, and Gertler (2000)	4	0.075	0.99	s	0.83	0.27

Note: The structural and policy parameters are given by the "Pre-Volcker (Prior I)" in Table 3 of Lubik and Schorfheide (2004), Section 1.5 of McCallum and Nelson (1999), and the baseline result in Table II of Clarida, Gali, and Gertler (2000).

Table 2: Eigenvalues

		Eigenvalues	
Lubik and Schorfheide (2004)	B_c	1.1767	0.5513
McCallum and Nelson (1999)	B_f	1.0640	0.9045
Clarida, Gali, and Gertler (2000)	B_s	1.0357	0.4595

us calibrate the effect of the expectational frictions (here, heterogeneity $1-\rho$) on stability conditions.

Table 2 shows the eigenvalues of coefficient matrices (B_c, B_f, B_s) using the calibrated parameters. All of the cases satisfy the stationary condition (8) so that these calibrated models are capable of exhibiting plausible sunspot equilibria of CF representation.

Figure 3 shows calibrated stability conditions under the contemporaneous rule and the different degrees of information heterogeneity $1-\rho$. The parameters of Lubik and Schorfheide (2004) are used, and their estimates of (ϕ_π, ϕ_x) are also shown.

The figure shows that the expectational frictions have a significant effect on the stability of sunspot equilibria. The region of stability is expanded dramatically by an increase in the degree of imperfection. This suggests that given a policy rule, even a small change in the information

eron (2007) estimate the serial correlation to be $0.87 \sim 0.95$ using US data during 1960-2002. Feve, Matheron, and Poilly (2007) obtain similar results using European data during 1987-2004.

structure can be a crucial reason for the outset of self-fulfilling fluctuations. Past boom-bust cycles might be explained by structural shocks that temporarily changed information structures.

In addition, those frictions well contribute to improving the plausibility of calibrated NK models with sunspot equilibria. In the absence of imperfection ($1 - \rho = 0$), sunspot equilibria are unstable under most passive rules, and the Lubik and Schorfheide (2004)'s estimates are out of the region of stability. These results reject the plausibility of their model. But, this implausibility is perfectly eliminated when $\rho \leq 0.5$ (that is, $1 - \rho \geq 0.5$).[28] Thus, the expectational frictions can resolve the stability puzzle that has been open in the learning literature.

These results are robust in the other calibrated NK models. In our supplementary analysis, stability conditions in the McCallum and Nelson (1999)'s model with the forward-looking rule and in the Clarida, Gali, and Gertler (2000)'s model with the semi-forward-looking rule are also calibrated. Their results are similar to above.[29]

In total, our calibrations confirm that the expectational frictions caused by imperfect information provide the significant effect in popular calibrated NK models. In the presence of imperfect information, the Taylor principle is important to prevent self-fulfilling fluctuations. This result is independent of the type of a policy rule.

6 Conclusion

This paper investigated the stability of sunspot equilibria under adaptive learning when agents' information sets of exogenous variables are imperfect. Sunspot-driven business cycle models are popular tools to account for non-fundamental business cycle fluctuations, and the recent global financial turmoil is described as arising from self-fulfilling fluctuations

[28] Bhattacharjee and Holly (2015, Fig. 1) estimate the interactions of the board members of the central bank using the data of the Bank of England. The average of the estimated coefficients of the regressions between policy decisions of the members is 0.32, which may be close to correlations of their preferences.

[29] In contrast to the Lubik and Schorfheide (2004)'s results, the estimates (ϕ_π, ϕ_x) in these two models are always in the regions of stability. Regarding Clarida, Gali, and Gertler (2000), Honkapohja and Mitra (2004) show that sunspot equilibria are unstable in their calibrated model.

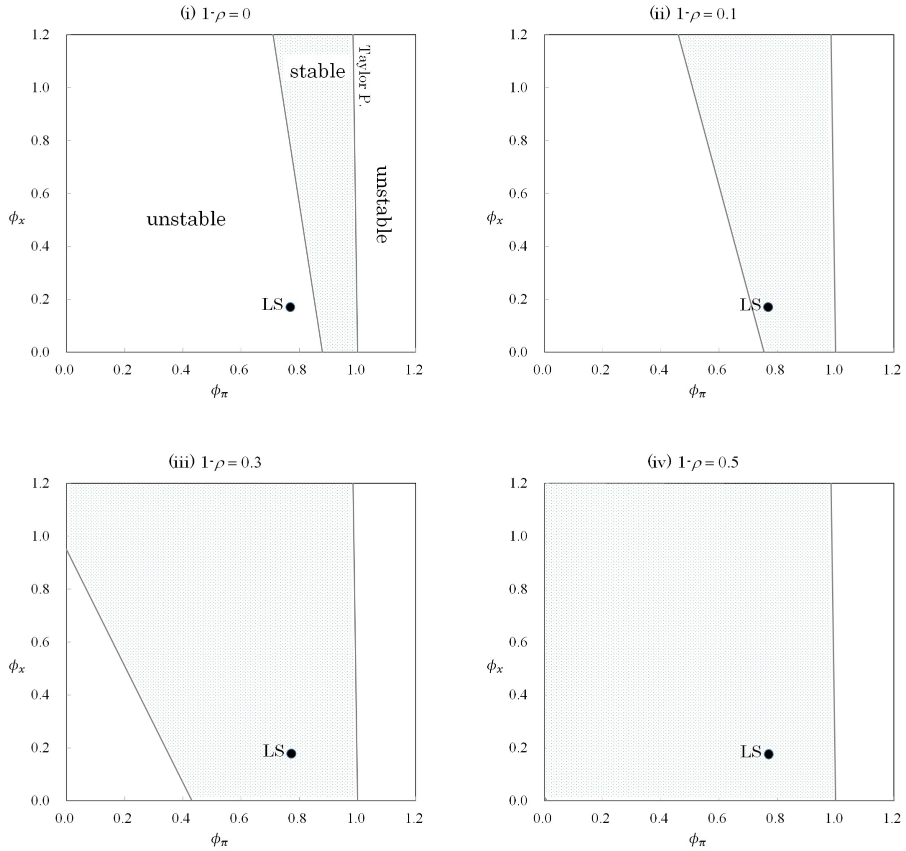

Figure 3: Stability conditions under the contemporaneous rule (Parameters: Lubik and Schorfheide, 2004).

based on the imperfect information of fundamental variables. While several studies have investigated the dynamics of sunspot equilibria in the learning framework, it has not been fully clarified how imperfect information affects the stability of sunspot equilibria. This study fills this gap, and using the NK model, examines whether imperfect information provides the stability conditions that are consistent with the empirical evidence.

The paper finds that imperfect information under learning creates expectational frictions that make the updating of expectations easy to converge. The expectational frictions have a positive effect on the stability of sunspot equilibria. Specifically, their stability conditions are relaxed as the degree of information imperfection (or the magnitude of the frictions) increases. In the NK model, expectational frictions provide the stability condition that is consistent with the empirical evidence: Sunspot equilibria are stable if and only if the Taylor principle is violated. These results suggest that those frictions raise the possibility of self-fulfilling fluctuations. In addition, the frictions may contribute to establishing empirically plausible business cycle models, and the Taylor principle is essential for preventing non-fundamental fluctuations.

Future works are expected as follows. A technical work will generalize our model to one that includes lagged endogenous variables. Our model is purely forward-looking to obtain analytical results; however, most business cycle models are also backward-looking. While McGough and Nakagawa (2016) find numerically that the positive effect of imperfect information is robust to the backward-lookingness of models, it is also necessary to examine the robustness analytically.

A second work is expected to reexamine stabilization policies provided by the existing learning literature. For example, several studies clarify the mechanism of a liquidity trap, its stability, and optimal policies to escape from it. However, our results imply that those policies could be ineffective in the presence of imperfect information that improves the stability of self-fulfilling fluctuations. In that case, it would be necessary to propose more stringent policies to escape the trap.

The last possible work is to account for the mechanism of past non-fundamental fluctuations that might have been driven by imperfect information. Many studies are seen in the RE literature particularly after the recent financial turmoil, but not enough in the learning literature. To this background, this study suggests in the learning framework that imperfect information under learning creates expectational frictions that affect the dynamics of self-fulfilling fluctuations. Using this mechanism, we could discover why the past non-fundamental fluctuations were observed after there occurred, for example, technological or financial innovations, which were initially unfamiliar to people. Further, we could account for past boom-bust cycles by relating them with the time-variation in the information structure.

7 Appendix

A Regularity Assumptions

Assumption 2

1. $\det(B) \neq 0$, $\det(I_m - B) \neq 0$, *and* $\det(I_{mn} - \Phi \otimes B) \neq 0$.
2. Φ *is a diagonal and nonnegative matrix whose diagonal elements exist in the interval* $[0, 1)$.
3. Γ *is a nonnegative matrix, in which* $0 \leq \rho_{ij} \leq 1$ *for each* $i, j \in \{1, ..., n\}$.

Assumption 2.1 ensures the forward-lookingness of the model and prevents the possibility that the non-explosive fundamental REE could be indeterminate (see Honkapohja and Mitra, 2006, Proposition 1).

The diagonal representation of Φ in Assumption 2.2 simplifies the analysis by equating the eigenvalues of Φ with its diagonal elements existing in the interval $[0, 1)$. Note that this assumption is not crucial for our analysis, because even if Φ were originally nondiagonal, Eq. (1) could be transformed to an equation that includes a diagonal autocorrelation matrix by premultiplying Eq. (1) by the $n \times n$ matrix formed from the eigenvectors of Φ. The diagonal elements in the interval $[0, 1)$ ensure the stationarity of w_t.

Neither is Assumption 2.3 crucial for our analysis, because any linear model can be transformed to the system with $\Gamma \geq 0$. For example, if any ρ_{ij} is negative in an original model, this negative correlation can be transformed to be positive by changing the sign of w_i (or w_j) and redefining the correlation between $-w_i$ and w_j as $\rho_{ij} \geq 0$. Applying this transformation to all negative correlations, the original model is transformed to the system with $\Gamma \geq 0$.

B Derivation of ODE under heterogeneously misspecified learning

The ODE under heterogeneously misspecified learning is obtained by accommodating the global convergence of the ODE associated with an RPE in Evans and Honkapohja (2001, Section 13.1.1). Agent i for each $i \in \{1,\ldots,n\}$ forms $E^*_{it}y_{t+1}$ by using real-time learning with the PLM (15) and the information set $\{y_s, w_{is}, d_s\}_{s=1}^{t}$. We assume the *t-dating of expectations* considered by Evans and Honkapohja (2001, chapter 10): coefficient parameters ϕ_{it} at time t are estimated with past data up to time $t-1$, $\{y_s, w_{is}, d_s\}_{s=1}^{t-1}$, and $E^*_{it}y_{t+1}$ is formed with ϕ_{it} and the contemporaneous data $\{y_t, w_{it}, d_t\}$. The estimates of the coefficient parameters $\phi'_{it} = (c_{it}, d_{it})$ are given by the least-squares projection of y_{t-1} on $z'_{i,t-1} = \left(w_{i,t-1}, \xi_{t-1}\right)$: $Ez_{i,t-1}\left(y_{t-1} - \phi'_{it}z_{i,t-1}\right)' = 0$. Then, the updating rule of ϕ_{it} is shown by the RLS representation:

$$\begin{aligned} \phi_{it} &= \phi_{i,t-1} + t^{-1}R_{it}^{-1}z_{i,t-1}\left(y_{t-1} - \phi'_{i,t-1}z_{i,t-1}\right)', && \text{(B.1)} \\ R_{it} &= R_{i,t-1} + t^{-1}\left(z_{i,t-1}z'_{i,t-1} - R_{i,t-1}\right), && \text{(B.2)} \end{aligned}$$

where $R_{it} = t^{-1}\sum_{s=1}^{t} z_{i,s-1}z'_{i,s-1}$, which is the updating of the matrix of the second moment of z_{it}.

The stochastic recursive algorithm (SRA) for ϕ_{it} for each i is obtained by substituting the ALM (18) into Eq. (B.1):

$$\phi_{it} = \phi_{i,t-1} + t^{-1}R_{it}^{-1}z_{i,t-1}\left(\begin{pmatrix} w'_{t-1} & \xi_{t-1} \end{pmatrix}\begin{pmatrix} S_{c,t-1} & d_{t-1} \end{pmatrix}' - z'_{i,t-1}\phi_{i,t-1}\right),$$

where we denote $S_{ct} \equiv Bc_t\Phi + C$, $c_t \equiv \frac{1}{n}(c_{1t},...,c_{nt})$, and $d_t \equiv \frac{1}{n}\sum_{i=1}^{n} d_{it}$.

To obtain the ODEs for ϕ_i associated with the SRA, we have to calculate the unconditional expectations of the updating terms in the SRA. The convergence of the SRA is analyzed by Marcet and Sargent (1989b) in the stochastic approximation approach, which is also introduced by Evans and Honkapohja (2001, chapter 6). Denote the operator E as the expectation of variables for ϕ_i fixed, taken over the invariant distributions of w_t. Then, by letting $Ez_iz'_j = \lim_{t\to\infty} Ez_{it}z'_{jt}$ for each $i, j \in \{1,...,n\}$, the unconditional expectation of the updating term in Eq. (B.1) is transformed to

$$\begin{aligned} &E\left[R_{it}^{-1}z_{i,t-1}\left(\begin{pmatrix} w'_{t-1} & \xi_{t-1} \end{pmatrix}\begin{pmatrix} S_{c,t-1} & d_{t-1} \end{pmatrix}' - z'_{i,t-1}\phi_{i,t-1}\right)\right] \\ &\quad = R_i^{-1}\left(Ez_iz'_i\right)\left[\begin{pmatrix} 0 \\ d \end{pmatrix} + \left(Ez_iz'_i\right)^{-1}\begin{pmatrix} \omega_{i1} & \cdots & \omega_{in} \\ 0 & \cdots & 0 \end{pmatrix} S'_c - \phi_{i,t-1}\right], \end{aligned}$$

where $S_c \equiv Bc\Phi + C$, $c \equiv \frac{1}{n}(c_1,...,c_n)$, $d \equiv \frac{1}{n}\sum_{i=1}^{n} d_i$, ω_{ij} is the covariance of w_i and w_j such that $\omega_{ij} \equiv \sigma_{ii}\rho_{ij}\sigma_{jj}$ for each i, j. When ζ denotes the variance of ξ_t, $Ez_iz'_i = \begin{pmatrix} \omega_{ii} & 0 \\ 0 & \zeta \end{pmatrix}$ because of the assumption $E(w_t\xi_t) = 0$. Next, the expectation of the updating term in Eq. (B.2) is given by $Ez_iz_i' - R_i$. Hence, the ODEs for ϕ_i and R_i associated with the SRA are obtained as

$$\frac{d\phi_i}{d\tau} = R_i^{-1}\left(Ez_iz_i'\right)\left(T\left(\phi_i\right)' - \phi_i'\right)', \tag{B.3}$$

$$\frac{dR_i}{d\tau} = Ez_iz_i' - R_i, \tag{B.4}$$

where

$$T\left(\phi_i\right)' \equiv \begin{pmatrix} \omega_{ii}^{-1}\begin{pmatrix} \omega_{i1} & \cdots & \omega_{in} \end{pmatrix} S_c' & d \end{pmatrix}.$$

Furthermore, because R_i and Ez_iz_i' in Eq. (B.4) are asymptotically equal, $R_i^{-1}\left(Ez_iz_i'\right)$ in Eq. (B.3) globally converges to unity. Hence, the stability of the ODE for $\phi_i' = (c_i, d_i)$ in Eq. (B.3) is determined by smaller differential equations:

$$\frac{d\phi_i}{d\tau} = T\left(\phi_i\right) - \phi_i. \tag{B.5}$$

In the same manner, smaller ODEs for the parameters $\left\{\phi_j\right\}_{j\neq i}^{n}$ are obtained.

The ODEs (B.5) for all i are represented by the ODEs for the aggregate parameters (c, d) in Eq. (19) as follows. First, the ODEs for all d_is have the same form, and d is an arithmetic average of all d_is. Then, the convergence property of d is equivalent to that of d_i for each i; the ODEs for all d_is are represented by a single ODE for d that has the same form as that for d_i:

$$\frac{dd}{d\tau} = T_d\left(d\right) - d,$$

where $T_d\left(d\right) \equiv S_d$. Next, the ODEs for all c_is are represented by a single ODE for the aggregate parameter c. If the ODEs of c_i in Eq.(B.5) for all i are multiplied by $\frac{1}{n}$ and combined in a single $1 \times n$ matrix, the single ODE for c is obtained by:

$$\frac{dc}{d\tau} = T_c\left(c\right) - c,$$

where

$$\begin{aligned} T_c\left(c\right) &\equiv \begin{pmatrix} \frac{1}{n}\omega_{11}^{-1}\begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \end{pmatrix} S_c' & \cdots & \frac{1}{n}\omega_{nn}^{-1}\begin{pmatrix} \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} S_c' \end{pmatrix}' \\ &= \left(Bc\Phi + C\right)\left(\frac{1}{n}\Psi\right), \end{aligned}$$

and

$$\begin{aligned} \Psi &\equiv \begin{pmatrix} 1 & \omega_{12}\omega_{22}^{-1} & \cdots & \omega_{1n}\omega_{nn}^{-1} \\ \omega_{21}\omega_{11}^{-1} & 1 & \cdots & \omega_{2n}\omega_{nn}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1}\omega_{11}^{-1} & \omega_{n2}\omega_{22}^{-1} & \cdots & 1 \end{pmatrix} \\ &= diag\left(\sigma_{ii}\right)_{1\leq i\leq n} \cdot \Gamma \cdot diag\left(\sigma_{ii}\right)_{1\leq i\leq n}^{-1}. \end{aligned}$$

The derivation is complete.

C Proof of Proposition 5

Consider the NK model (23)–(24) and the contemporaneous rule (25), and let us obtain the parameter region for matrix B_c to satisfy the stationary condition (8) and the stability condition (21).

The NK model with the contemporaneous rule is represented in the following form:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = B_c E_t \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} - \begin{bmatrix} 1+\alpha\phi_x & \alpha\phi_\pi \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} \alpha \\ 0 \end{bmatrix} w_t,$$

where

$$B_c \equiv \begin{bmatrix} 1+\alpha\phi_x & \alpha\phi_\pi \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & \alpha \\ 0 & \beta \end{bmatrix}.$$

First, the complement of the stationary condition (8) is $\lambda\left[B_c - I_2\right] \leq 0$: $\operatorname{tr}\left(B_c - I_2\right) \leq 0$ and $\det\left(B_c - I_2\right) \geq 0$. That is,

$$\begin{aligned} \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\beta\right) &\geq 0, && \text{(C.1)} \\ \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\beta\right) &\geq -\frac{1-\beta+\alpha\kappa+\alpha\beta\phi_x}{2\alpha}. && \text{(C.2)} \end{aligned}$$

Eq. (C.2) is redundant by Eq. (C.1) and $1-\beta > 0$. Thus, the stationary condition is

$$\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\beta\right) < 0. \qquad \text{(C.3)}$$

Note that in this region, the eigenvalues of matrix B_c are all real as they are

$$\frac{1}{2\left(\alpha\phi_x + \alpha\kappa\phi_\pi + 1\right)} \times \left(\left(\alpha\kappa + \alpha\beta\phi_x + 1 + \beta\right) \pm \sqrt{\left(\alpha\kappa + \alpha\beta\phi_x + 1 - \beta\right)^2 - 4\alpha\beta\left(\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\beta\right)\right)}\right).$$

Next, the stability condition (21) is $\lambda\left[B_c - \mu\lambda\left[\Phi\right]^{-1} I_2\right] < 0$: $\operatorname{tr}\left(B_c - \mu\lambda\left[\Phi\right]^{-1} I_2\right) < 0$ and $\det\left(B_c - \mu\lambda\left[\Phi\right]^{-1} I_2\right) > 0$. Both conditions provide

$$\begin{aligned} \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\beta\right) &> -\frac{1}{\alpha}\left(1-\mu^{-1}\lambda\left[\Phi\right]\right)\left(1-\beta\mu^{-1}\lambda\left[\Phi\right] + \alpha\kappa + \alpha\beta\phi_x\right) && \text{(C.4)} \\ &\quad -\frac{\mu^{-1}\lambda\left[\Phi\right]}{2\alpha}\left(1-\beta\mu^{-1}\lambda\left[\Phi\right] + \alpha\kappa + \alpha\beta\phi_x + \beta\left(1-\mu^{-1}\lambda\left[\Phi\right]\right)\right), \\ \kappa\left(\phi_\pi - 1\right) + \phi_x\left(1-\beta\right) &> -\frac{1}{\alpha}\left(1-\mu^{-1}\lambda\left[\Phi\right]\right)\left(1-\beta\mu^{-1}\lambda\left[\Phi\right] + \alpha\kappa + \alpha\beta\phi_x\right). && \text{(C.5)} \end{aligned}$$

Eq. (C.4) is redundant by Eq. (C.5) and $\mu\lambda\left[\Phi\right]^{-1} > 1$.

Therefore, the combination of Eqs. (C.3) and (C.5) provides Eq. (29).

D Proof of Proposition 6

Consider the NK model (23)–(24) and the forward-looking rule (26), and let us obtain the parameter region for matrix B_f to satisfy the stationary condition (8) and the stability condition (21).

The NK model with the forward-looking rule is represented in the following form:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = B_f E_t \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} - \alpha \begin{bmatrix} 1 \\ \kappa \end{bmatrix} w_t,$$

where

$$B_f \equiv \begin{bmatrix} 1 & 0 \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1-\alpha\phi_x & -\alpha(\phi_\pi - 1) \\ 0 & \beta \end{bmatrix}.$$

First, the complement of the stationary condition (8) is $\lambda[B_f - I_2] \leq 0$: $\operatorname{tr}(B_f - I_2) \leq 0$ and $\det(B_f - I_2) \geq 0$. That is,

$$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) \geq -\frac{1-\beta+\alpha\beta\phi_x}{\alpha}, \tag{D.1}$$

$$\phi_x(1-\beta) + \kappa(\phi_\pi - 1) \geq 0. \tag{D.2}$$

Eq. (D.1) is redundant by Eq. (D.2) and $1-\beta > 0$. Thus, the stationary condition is

$$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) < 0. \tag{D.3}$$

Note that in this region, the eigenvalues of matrix B_f are all real as they are

$$\frac{1}{2}\left(1+\beta-\alpha\kappa(\phi_\pi - 1) - \alpha\phi_x\right)$$
$$\pm\frac{1}{2}\sqrt{\left(1-\beta+\alpha\kappa(\phi_\pi-1)+\alpha\phi_x\right)^2 - 4\alpha\left(\kappa(\phi_\pi - 1) + \phi_x(1-\beta)\right)}.$$

Next, the stability condition (21) is $\lambda[B_f] < \mu\lambda[\Phi]^{-1}$: $\operatorname{tr}\left(B_f - \mu\lambda[\Phi]^{-1} I_2\right) < 0$ and $\det\left(B_f - \mu\lambda[\Phi]^{-1} I_2\right) > 0$. Both conditions provide

$$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) > -\left(1-\mu^{-1}\lambda[\Phi]\right)\frac{1-\beta(1-\alpha\phi_x)\mu^{-1}\lambda[\Phi]}{\alpha\mu^{-1}\lambda[\Phi]} \tag{D.4}$$

$$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) > -\frac{1-\beta(1-\alpha\phi_x)\mu^{-1}\lambda[\Phi]}{\alpha\mu^{-1}\lambda[\Phi]} - \frac{1-\mu^{-1}\lambda[\Phi]}{\alpha\mu^{-1}\lambda[\Phi]} \tag{D.5}$$

Eq. (D.5) is redundant by Eq. (D.4) and $\mu\lambda[\Phi]^{-1} > 1$.

Therefore, the combination of Eqs. (D.3) and (D.4) provides Eq. (30).

E Proof of Proposition 7

Consider the NK model (23)–(24) and the semi-forward-looking rule (27), and let us obtain the parameter region for matrix B_s to satisfy the stationary condition (8) and the stability condition (21).

The NK model with the semi-forward-looking rule is represented in the following form:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = B_s E_t \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} - \alpha \begin{bmatrix} 1+\alpha\phi_x & 0 \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \kappa \end{bmatrix} w_t,$$

where

$$B_s \equiv \begin{bmatrix} 1+\alpha\phi_x & 0 \\ -\kappa & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -\alpha(\phi_\pi - 1) \\ 0 & \beta \end{bmatrix}.$$

First, the complement of the stationary condition (8) is $\lambda[B_s - I_2] \leq 0$: $\operatorname{tr}(B_s - I_2) \leq 0$ and $\det(B_s - I_2) \geq 0$. That is,

$$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) \geq -\frac{1-\beta+\alpha\phi_x}{\alpha}, \tag{E.1}$$

$$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) \geq 0. \tag{E.2}$$

Eq. (E.1) is redundant by Eq. (E.2). Thus, the stationary condition is

$$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) < 0. \tag{E.3}$$

Note that in this region, the eigenvalues of matrix B_s are all real as they are

$$\frac{1}{2\alpha\phi_x + 2}\left((1+\beta+\alpha\phi_x) - \alpha(\kappa(\phi_\pi - 1) + \phi_x(1-\beta))\right)$$
$$\pm\frac{1}{2\alpha\phi_x + 2}\sqrt{((1+\beta+\alpha\phi_x) - \alpha(\kappa(\phi_\pi - 1) + \phi_x(1-\beta)))^2 - 4\beta(1+\alpha\phi_x)}.$$

Next, the stability condition (21) is $\lambda[B_s] < \mu\lambda[\Phi]^{-1}$: $\operatorname{tr}\left(B_s - \mu\lambda[\Phi]^{-1} I_2\right) < 0$ and $\det\left(B_s - \mu\lambda[\Phi]^{-1} I_2\right) > 0$. Both conditions provide

$$\kappa(\phi_\pi - 1) + \alpha\phi_x(1-\beta) > \alpha^{-1}\left(\beta - \left(2\mu\lambda[\Phi]^{-1} - 1\right)(1+\alpha\phi_x)\right), \tag{E.4}$$

$$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) > -\frac{\left(1-\mu^{-1}\lambda[\Phi]\right)\left(1-\beta\mu^{-1}\lambda[\Phi] + \alpha\phi_x\right)}{\alpha\mu^{-1}\lambda[\Phi]} \tag{E.5}$$

Eq. (E.4) is redundant by Eq. (E.5) and $\mu\lambda[\Phi]^{-1} > 1$.

Therefore, the combination of Eqs. (E.3) and (E.5) provides Eq. (31).

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