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The resource allocation problems with interpersonal comparisons of welfare: An axiomatization of the impartial Walras rule*

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Abstract

The extended sympathy approach, which has been studied so far in the abstract framework of social choice, is applied to the resource allocation problem of exchange economies with a finite number of agents and goods.

The central issue in this study is the axiomatic analysis of the *impartial Walras rule* that associates with each preference profile the union of the sets of Walrasian allocations operated from permuted initial endowments, where the union is over all permutations.

We show that the impartial Walras rule is the unique rule that satisfies Suppes nondiscrimination, Suppes rationality, Suppes equity, and local independence.

Keywords:extended sympathy;Suppes criterion;the Walras rule;local independence;cardinality and noncomparability;interpersonal comparisons of welfare

1 Introduction

The notion of extended preference makes it possible to compare the welfare of different individuals in social choice. It has been developed primarily in the literature of Arrow's impossibility theorem, in the abstract framework of social

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choice.[1] The purpose of this study is to apply this approach to the resource allocation problem of exchange economies with a finite number of agents and goods.

The central issue is an axiomatic analysis of the *impartial Walras rule*. This rule associates with each preference profile the union of the sets of Walrasian allocations operated from permuted initial endowments, where the union is over all permutations. The main result is Theorem 1, which says that the impartial Walras rule is the unique rule satisfying Suppes-nondiscrimination (SN), Suppes-rationality (SR), Suppes-equity (SE), and local independence (LI).

SN is a generalization of anonymity or nondiscrimination and says that a rule should treat equally in social choice any two allocations that are supposed to be identical through permuting agents. SR is a generalization of individual rationality and gives each agent a utility level that is not lower than the utility attained through a permutation of agents. SE is an interpersonal extension of Pareto optimality. LI, as propose by Nagahisa (1991), is a requirement of informational economization. Despite the formulation defined only with subjective preferences, we discuss that LI has the meaning of interpersonal comparisons of welfare.

Theorem 1 is the counterpart of the main result of Nagahisa (1991), an axiomatization of the Walras rule.[2] If interpersonal comparisons of welfare are allowed, the similar axioms as those in Nagahisa (1991) lead us to the impartial Walras rule, not to the Walras rule. Theorem 2 shows that another definition of rules provides an alternative axiomatization of the impartial Walras rule. It is equivalent to Theorem 1 despite the difference in the definition of rules.

[1] d'Aspremont (1985), d'Aspremont and Gevers (2002), Sen (1970,1977,1986), and Suzumura (1983) surveyed this area. Blackorby et al. (1984) provide a diagrammatic introduction. The most recent contribution is Yamamura (2017).

The exceptions are Sen (1974a,b) and Deschamps and Gevers (1978), who consider the income distribution problem in a single commodity economy.

[2] Gevers (1986) and Hurwicz (1979) initiated the study of the axiomatic analysis of the Walras rule. Hammond (2010) is a comprehensive survey of the field.

We describe the impartial Walras rule as a rule chosen in a fictional arena of social choice. An example is an original position, assumed by Rawls (1971), covered with the veil of ignorance. A position assumed by Hare (1981) and Harsanyi (1955) where each agent replaces the other's position equally is also another example.

We point out that if each agent has an identical endowment, Theorem 1 provides an alternative axiomatization of the equal-income Walras rule studied by Thomson (1988), Nagahisa and Suh (1995), Maniquet (1996), and Toda (2004).

This study is organized as follows. We provide notation and definitions in Section 2. We state the main results in Sections 3 and 4 and prove them in Section 5. Section 6 is the conclusion.

2 Notation and Definitions

2.1 Exchange Economies

We consider exchange economies with a finite number of agents and a finite number of private goods. Let $N = \{1, 2, ..., n\}$ and $L = \{1, 2, ..., l\}$ be the set of agents and the set of private goods, respectively. All agents have the same consumption set R^l_+. Let $z_i = (z_{i1}, ..., z_{il}) \in R^l_+$ and $z = (z_1, ..., z_n) \in R^{nl}_+$ be agent i's consumption and an allocation respectively. Let $\omega_i \in R^l_{++}$ be agent i's initial endowment, fixed throughout the paper. Let $\omega = (\omega_1, ..., \omega_n)$. An allocation z is feasible if $\sum_{i \in N} z_i = \sum_{i \in N} \omega_i$. Let Z be the set of feasible allocations.

Let $\succcurlyeq_i$ be agent i's preference on R^l_+. A profile $\succcurlyeq = (\succcurlyeq_i)_{i \in N}$ is a list of the preferences. Let $\Delta^l := \{p \in R^l_+ : \sum_{i=1}^{l} p_i = 1\}$ and $int.\Delta^l := \{p \in R^l_{++} : \sum_{i=1}^{l} p_i = 1\}$. Let IL be the set of profiles such that $\succcurlyeq = (\succcurlyeq_i)_{i \in N} \in IL$ if and only if there exists some $p \in int.\Delta^l$ such that for each $\succcurlyeq_i$, $x \succcurlyeq_i y \Longleftrightarrow px \geq py$ for all $x, y \in R^l_+$. A profile $\succcurlyeq$ is in IL if and only if every agent has the same preference represented by a linear utility function. We use the notation $\succsim^p = (\succsim_i^p)_{i\in N}$ if we need to specify p. Let Q be the set of preferences satisfying (i)-(iii). (i) $\succsim_i$ is continuous and convex on R^l_+ and continuously differentiable on R^l_{++}. (ii) $x \geq y \& x \neq y$ implies $x \succsim_i y$, and if besides $x \in R^l_{++}$, this implies $x \succ_i y$ (monotonicity). (iii) for any $x \in R^l_{++}$, $\{y \in R^l_+ : y \succsim_i x\} \subset R^l_{++}$ (boundary condition). Let $Q^n = \overbrace{Q \times \cdots \times Q}^{n}$. Let $D = IL \cup Q^n$ be the domain, which is the same as that of Nagahisa (1991).[3] Note that as for $\succsim_i \in Q$, $z_i \sim_i 0$ for any $z_i \notin R^l_{++}$.

Given a profile, the Pareto optimal, individually rational, and Walrasian allocations are defined as usual: (i) $z \in Z$ is Pareto optimal if and only if there is no feasible allocation z' such that $z'_i \succsim_i z_i$ for all $i \in N$ and $z'_i \succ_i z_i$ for some $i \in N$: (ii) $z \in Z$ is individually rational if and only if $z_i \succsim_i \omega_i$ for all $i \in N$:(iii) $z \in Z$ is a Walrasian allocation if and only if there is a price vector $p \in int.\Delta^l$ such that for all $i \in N$, $z_i \succsim_i x_i$ for all $x_i \in R^l_+$ such that $px_i \leq p\omega_i$. Let $PO(\succsim)$ be the set of the Pareto optimal allocations. We read $IR(\succsim)$ and $W(\succsim)$ in the same way. We occasionally use the notation $W(\succsim, \omega)$ and $IR(\succsim, \omega)$ instead of $W(\succsim)$ and $IR(\succsim)$.

2.2 Extended Preferences

The notion of *extended preferences* is based on the principle of extended sympathy mentioned by Arrow (1963) and initiated by Suppes (1966) and Sen (1970). The basic idea is that a hypothetically existing ethical observer compares the welfare of different persons from a social point of view while respecting (or sympathizing with) their subjective preferences. An extended preference $\succsim_E$ from $\succsim \in D$ is a complete and transitive binary relation on $R^l_+ \times N$. We read $(x, i) \succsim_E (y, j)$ as "being agent i with consumption x is at least as well off as

[3] The domain Nagahisa and Suh (1995) employed is more natural. However, we prefer mathematical tractability here.

being agent j with consumption y."[4] We read $\succ_E$ and $\sim_E$ as usual. We assume the axiom of identity, which says that for any $i \in N$, the restriction of $\succsim_E$ to $R^l_+ \times \{i\}$ is identical to $\succsim_i$; $x \succsim_i y \Longleftrightarrow (x,i) \succsim_E (y,i)$ for any $x, y \in R^l_+$ and any $i \in N$.[5]

The two extended preferences illustrated in the examples below play an important role in subsequent sections.

Example 1 Let $\succsim \in D$ and $p \in int.\Delta^l$ be given. The extended preference $\succsim_{E(p)}$ is defined by

$$(x,i) \succsim_{E(p)} (y,j) \Longleftrightarrow \min\{pq : q \sim_i x\} \geq \min\{pq : q \sim_j y\}.^6$$

We regard p as prices. In $\succsim_{E(p)}$, the prices p works as an indicator to compare the welfare of different agents. We compare the minimum amount of expense that agent i needs to spend to achieve the level of utility at x with that of agent j's y. In comparison, $\succsim_{E(p)}$ prefers more to less.

If $\succsim^p \in IL$, then $\succsim^p_{E(p)}$ reduces to a simple form as follows.

Example 2 $(x,i) \succsim^p_E (y,j) \Longleftrightarrow px \geq py.$

If we need to refer to multiple extended preferences from $\succsim$, we use the notation $\succsim_{E'}$, $\succsim_{E''}$ and so on. Note the difference between $\succsim'_E$ and $\succsim_{E'}$.

2.3 Suppes criterion

Let $\succsim_E$ be taken arbitrarily and fixed throughout this subsection. Given an allocation z and a permutation π on N, let z^π be an allocation such that $z^\pi_i = z_{\pi(i)}$ for every $i \in N$. Let Π be the set of permutations. The three relations below constitute the *Suppes criterion* (Suppes 1966), occasionally called the grading principle, interpreted as an interpersonal extension of the Pareto criterion.

[4] Note that we admit interpersonal comparisons of welfare, but still deny cardinality.

[5] Almost all literature related to extended preferences assumes this axiom. Refer to Sen (1970) and d'Aspremont (1985) for more details.

[6] Note that $\min\{pq : q \sim_i x\} = 0$ if $\succsim_i \in Q$ and $x \notin R^l_{++}$.

Given allocations z, z', z is at least as just as z' if there is some $\pi \in \Pi$ such that $(z_i, i) \succcurlyeq_E (z'_{\pi(i)}, \pi(i))$ for all $i \in N$.

If this holds with $\succ_E$ for at least one member i, z is more just than z'. In contrast, if that holds with $\sim_E$ for all i, z is equally as just as z'. All are well defined and transitive. The relation of "equally as just as" is symmetric whereas that of "more just than" is asymmetric.

A feasible allocation z is Suppes-equitable if and only if there is no feasible allocation z' that is more just than z. Let $SE(\succcurlyeq_E)$ be the set of Suppes-equitable allocations. A feasible allocation z is Suppes-rational if and only if it is at least as just as ω. Let $SR(\succcurlyeq_E)$ be the set of Suppes-rational allocations. Note that $SE(\succcurlyeq_E) \subset PO(\succcurlyeq)$ and $IR(\succcurlyeq) \subset SR(\succcurlyeq_E)$ because of the axiom of identity. Note also that $SE(\succcurlyeq_E)$ is nonempty if there exist utility functions u_i ($i \in N$) such that $(x, i) \succcurlyeq_E (y, j) \iff u_i(x) \geq u_j(y)$ for all $i, j \in N$ and all $x, y \in R^l_+$, as feasible allocations maximizing the sum of the utilities on Z are Suppes-equitable. Thus $SE(\succcurlyeq_{E(p)})$ is nonempty.

2.4 Rules

Let $F : D \longrightarrow Z$ be a *social choice rule*, a *rule* hereafter, which associates with each profile a nonempty subset of feasible allocations. A rule F decides $F(\succcurlyeq)$ through interpersonal comparisons of welfare. It uses *some* extended preferences generated from $\succcurlyeq$, but not necessarily *all*.

Given a profile $\succcurlyeq \in D$, let $D(\succcurlyeq)$ be a nonempty subset of $\succcurlyeq_E$. A rule F decides $F(\succcurlyeq)$ based only on extended preferences in $D(\succcurlyeq)$, not on extended preferences excluded from $D(\succcurlyeq)$. Let us call $\underset{\succcurlyeq \in D}{\cup} D(\succcurlyeq)$ the *extended domain.* To simplify the notation, we occasionally write $\mathcal{D} = \bigcup_{\succcurlyeq \in D} D(\succcurlyeq)$. We consider the following four assumptions.

D.1. For any $\succcurlyeq^p \in IL$, then $D(\succcurlyeq^p) = \{\succcurlyeq^p_E\}$.

D.2. $(x, i) \succcurlyeq_E (0, j)$ for all $i, j \in N$ and all $x \in R^l_+$.

D.3. For any $\succsim \in D$ and any $p \in int.\Delta^l$, $\succsim_{E(p)} \in D(\succsim)$.

We say that a set of utility functions $(u_i)_{i\in N}$ represents $\succsim_E$ if $(x,i) \succsim_E (y,j) \Longleftrightarrow u_i(x) \geq u_j(y)$ for all $i,j \in N$ and all $x,y \in R^l_+$.

D.4.For any $\succsim \in D$ and any $\succsim_E \in D(\succsim)$, there exists a set of utility functions $(u_i)_{i\in N}$ represents $\succsim_E$.

There exist many extended domains satisfying the four conditions. The smallest one is $D(\succsim) = \{\succsim_{E(p)} : p \in int.\Delta^l\}$ for any $\succsim \in D$. The largest one is as follows. Keeping D.1, and for any $\succsim \in Q^n$, we define $D(\succsim)$ as the set of all extended preferences obtained by comparison of utilities among agents subject to $u_i(0)$ being equal across all i, where u_i represents $\succsim_i$.

There is no compelling reason to dismiss D.1. If everyone has the same linear preference, that preference needs to be the extended preference, and no other extended preference is considered possible. It is hard to reject D.2, which reflects our intuition that as long as other conditions are equal, people without wealth are the most miserable in the world.

D.3 can be justified as follows. Let p be the supporting price of an allocation z. If we look only around z, we can think of $\succsim_{E(p)}$ as being identical to $\succsim^p_E$. The two extended preferences are locally identical around z. Let us request that the ethical observer only use the local information about preferences in this sense. Then, as long as $\succsim^p$ is allowed as an extended preference, so is $\succsim_{E(p)}$.

D.4 says that extended preferences should have their utility representations as well as individual preferences. Refer to Proposition 1, which shows that two additional assumptions of extended preferences assure D.4.

The Walras rule W and the impartial Walras rule IW are defined as follows.

$$W(\succsim) = W(\succsim, \omega) \text{ for any } \succsim \in D$$

$$IW(\succcurlyeq) = \underset{\pi\in\Pi}{\cup} W(\succcurlyeq, \omega^{\pi}) \text{ for any } \succcurlyeq\in D$$

The definition of the Walras rule is as usual. The impartial Walras associates with each profile the union of the sets of Walrasian allocations operated from permuted initial endowments, where the union is over all permutations.

The equal-income Walras rule is the Walras rule when every agent has the same initial endowment. Therefore, the impartial Walras rule reduces to the equal-income Walras rule in this case.

2.5 Axioms

Let F be a rule. F satisfies **Suppes-nondiscrimination** (**SN**) if $\forall \succcurlyeq\in D$, $\forall z, z' \in Z$, $z \in F(\succcurlyeq) \iff z' \in F(\succcurlyeq)$ if z and z' are equally as just as each other for any $\succcurlyeq_E\in D(\succcurlyeq)$. F satisfies **Suppes-equity** (**SE**) if $\forall \succcurlyeq\in D$, $F(\succcurlyeq) \subset \bigcup_{\succcurlyeq_E\in D(\succcurlyeq)} SE(\succcurlyeq_E)$. F satisfies **Suppes-rationality** (**SR**) if $\forall \succcurlyeq\in D$, $F(\succcurlyeq) \subset \bigcup_{\succcurlyeq_E\in D(\succcurlyeq)} SR(\succcurlyeq_E)$. Note that $\emptyset \neq PO(\succcurlyeq) \subset \bigcup_{p\in int.\Delta^l} SE(\succcurlyeq_{E(p)}) \overset{D.2}{\overset{\downarrow}{\subset}} \bigcup_{\succcurlyeq_E\in D(\succcurlyeq)} SE(\succcurlyeq_E)$.

SN says that if two feasible allocations are equally as just as each other for all extended preferences, a rule cannot deal with them differently.

SE and SR stand on a similar idea. For any allocation selected by a rule, SE (SR) requires at least an extended preference making it Suppes-equitable (Suppes-rational).

Given $\succcurlyeq\in D$ and $z_i \in R^l_{++}$, we say that $p \in int.\Delta^l$ is the supporting price of $\succcurlyeq$ at z_i if i's indifference curve passing through z_i is tangent to the line $\{x : px = pz_i\}$ at z_i. Given $\succcurlyeq\in D$ and $z \in R^{nl}_{++}$, we say that $p \in int.\Delta^l$ is the supporting price of $\succcurlyeq$ at z if for any i, $p \in int.\Delta^l$ is the supporting price of $\succcurlyeq_i$ at z_i.

Let $p(\succsim_i, z_i)$ be the supporting price of $\succsim$ at z_i. F satisfies **Local Independence** (**LI**) if for any $\succsim, \succsim' \in D$ and any $z \in Z \cap R^{nl}_{++}$, if $p(\succsim_i, z_i) = p(\succsim'_i, z_i)$ for all i, then $z \in F(\succsim) \iff z \in F(\succsim')$. LI, used in the axiomatization of the Walras rule by Nagahisa (1991), says that rules should use only the local information around z.[7]

Now, we discuss that LI compares the welfare of agents if $\succsim$ and $\succsim'$ share the same supporting price p at z. In the discussion, we remark two points below.

First, the interpersonal comparisons of welfare are carried out along a similar line as *cardinality and noncomparability* (CNC) (also called cardinal noncomparability, or cardinal measurability), an invariance axiom in the literature of social welfare functionals.[8] Second, D.1 is a clue in that comparison. We can regard every $\succsim_E \in D(\succsim)$ and every $\succsim'_E \in D(\succsim')$ as identical to $\succsim^p_E$.

Take $\succsim_E \in D(\succsim)$ and $\succsim'_E \in D(\succsim')$ arbitrarily. Refer to Lemma 3 in the next section, which says that under the supposition, for any $(u_i)_{i \in N}$ representing $\succsim_E$ and any $(u'_i)_{i \in N}$ representing $\succsim'_E$, the following relation holds: for any i, there exist some constants $\alpha_i, \beta_i, \alpha'_i, \beta'_i$ such that

$u_i(x_i) = \alpha_i p x_i + \beta_i + \varepsilon_i(x_i)$ for all x_i, where $\varepsilon_i(z_i) = 0$ and $\frac{\varepsilon_i(x_i)}{||x_i - z_i||} \longrightarrow 0$ as $x_i \longrightarrow z_i$; and

$u'_i(x_i) = \alpha'_i p x_i + \beta'_i + \varepsilon'_i(x_i)$ for all x_i, where $\varepsilon'_i(z_i) = 0$ and $\frac{\varepsilon'_i(x_i)}{||x_i - z_i||} \longrightarrow 0$ as $x_i \longrightarrow z_i$.

We can regard ε_i and ε'_i as higher-order error terms. When we focus on only around z, $\succsim_E$ is created by utility comparisons of $u_i(x_i) = \alpha_i p x_i + \beta_i$, which are positive affine transformations of $u_i(x_i) = p x_i$. The constants, $(\alpha_i)_{i \in N}$ and $(\beta_i)_{i \in N}$, are accessories, which facilitate considering $\succsim_E$ as identical to $\succsim^p_E$. The

[7] Refer to Nagahisa (1991) for more details on LI. Yoshihara (1998), Fleurbaey et al. (2005), and Miyagishima (2015) contribute generalizations and related axioms of LI. Sakai (2009) provides an ordering version of LI. Urai and Murakami (2015) use LI for economies with money.

[8] Refer to Bossert and Weymark (2004), d'Aspremont (1985), d'Aspremont and Gevers (1977), Fleurbaey and Hammond (2004), Mongin and d'Aspremont (2004), and Sen (1970) for more details on CNC.

same holds for $\succsim'_E$. Note also that this argument holds for *all* $\succsim_E$ and *all* $\succsim'_E$.

According to D.1, $D(\succsim^p)$ contains $\succsim^p_E$ only. In contrast, $D(\succsim)$ and $D(\succsim')$ in most cases contain more. Thus, there may exist $\succsim_E$ or $\succsim'_E$ that makes a different judgment from $\succsim^p_E$ around z in welfare comparison. For example, $(x, i) \succsim^p_E (y, j)$ and $(x, i) \prec_E (y, j)$, which is equal to $px_i \geq py_j$ and $\alpha_i px + \beta_i < \alpha_j px + \beta_j$. This difference arises because $\succsim_E$ uses information about preferences other than around z. Invoking Lemma 3 again, we know that the materials in creating $\succsim_E$ and $\succsim^p_E$ are identical as long as we look at them only around z. The difference in weights and scales are of no importance because it results from the overuse of the information of preferences. Conversely, we can say that despite the identity with the same linear preference, $\succsim_E$ treats agents as different persons with different weights and scales of utility, which is against the spirit of D.1. Ignoring that difference, we conclude that *every* $\succsim_E$ contains as much information as $\succsim^p_E$ does around z. Thus, deciding whether $z \in F(\succsim)$ or not is equivalent to whether $z \in F(\succsim^p)$ or not. The same is true between $\succsim^p$ and $\succsim'$. Thus we conclude $z \in F(\succsim) \Longleftrightarrow z \in F(\succsim')$, which is LI.

If we wish to formulate LI in terms of extended preferences, the following is an alternative. We say that $\succsim_E$ is *essentially-identical* to $\succsim^p_E$ around z if there exist a set of utility functions $(u_i)_{i \in N}$ such that (i) $(u_i)_{i \in N}$ represents $\succsim_E$, and (ii) for each i, there exists some constants $\alpha_i > 0$, β_i, and a higher-order error term ε_i such that $u_i(x_i) = \alpha_i px_i + \beta_i + \varepsilon_i(x)$ for all $x_i \in R^l_+$, where $\varepsilon_i(z_i) = 0$ and $\frac{\varepsilon_i(x_i)}{||x_i - z_i||} \longrightarrow 0$ as $x_i \longrightarrow z_i$. A rule F satisfies **Extended Local Independence** (**ELI**) if for any $\succsim \in D$ and any $z \in Z \cap R^{nl}_{++}$, if all $\succsim_E \in D(\succsim)$ are essentially-identical to $\succsim^p_E$ around z, then $z \in F(\succsim) \Longleftrightarrow z \in F(\succsim^p)$. We show that ELI is equivalent to LI if Pareto optimality (PO) is satisfied (Lemma 4). ELI says nothing in the case of no supporting price. However, this is enough for the axiomatization of the impartial Walras rule. No further discussion is necessary.

The above argument reminds us of local invariance, as proposed by Nagahisa (1991), which is equivalent to local independence. It requires that for any two profiles $\succcurlyeq, \succcurlyeq' \in D$, if there exist sets of utility functions $(u_i)_{i\in N}$ and $(u_i')_{i\in N}$ such that for any i, there exist constants, $\alpha_i > 0$, β_i, and a higher-order error term ε_i such that $u_i(x_i) = \alpha_i u_i'(x_i) + \beta_i + \varepsilon_i(x_i)$ for all x_i, then $z \in F(\succcurlyeq) \iff z \in F(\succcurlyeq')$. However, note a difference, which is that ELI is applicable only when $u_i'(x_i) = px_i$. The representation of $\succcurlyeq'$ by the same linear preference and applying D.1 are the core idea behind the argument.

Last but not least, we point out that for all the axioms except for LI, the strength depends on the size of $D(\succcurlyeq)$. The richer the $D(\succcurlyeq)$, the weaker the axioms.

3 Results

Theorem 1 *Assume D.1-D.4. The impartial Walras rule is the unique rule satisfying SN, SR, SE, and LI.*

The proof of Theorem 1 follows along a similar line as the axiomatization of the Walras rule (Nagahisa 1991). As SN and SE play the same role as those counterparts in Nagahisa, we conclude that SR is the most responsible for Theorem 1.

The idea of fictional social choice supports the impartial Walras rule. Two interpretations of "fictional" are open to us. One interpretation is to think of it as a social choice problem where no one knows who has which initial endowment.[9] This interpretation reminds us of the veil of ignorance (Rawls 1971). A logic similar to that of Rawls works with this interpretation. The impartial

[9] Speaking more accurately, we consider a problem where everyone knows the set of initial endowments, but no one knows who owns which of them.

Walras rule is unlikely to be supported if everyone knows their endowments perfectly. The Walras rule will be more favored in the state without the veil of ignorance. However, with the veil of ignorance, the impartial Walras rule can be justified.

A social choice problem in which everyone imagines to own another agent's initial endowment with equal probability can also be another interpretation. This interpretation is related to the idea, putting oneself in another's shoes, and is reminiscent of the suppositions of Hare (1981) and Harsanyi (1955). It is also semantically the same as the situation where everyone has the same initial endowment. In this case, the impartial Walras rule is equivalent to the equal-income Walras rule studied by Thomson (1988), Nagahisa and Suh (1995), Maniquet (1996), and Toda (2004). Thus Theorem 1 can be regarded as an axiomatization of the equal-income Walras rule in the case.

The four examples below illustrate that the axioms are independent and that Theorem 1 does not hold if one of the axioms lacks.

Example 3 (The impartial Walras rule operated from different endowments) Let ϖ be a new initial endowment such that $\varpi_i = \frac{\omega_i}{2}$ $(i = 1, ..., n-1)$ and $\varpi_n = \omega_n + \frac{1}{2}\sum_{i=1}^{n-1}\omega_i$. The rule F is given by $F(\succcurlyeq) = \bigcup_{\pi\in\Pi} IW(\succcurlyeq, \varpi^{\pi})$ for any $\succcurlyeq \in D$. This rule satisfies all the axioms except for SR.

Example 4 (The Walras rule) The Walras rule satisfies all the axioms except for SN.

Example 5 (The impartial Walras plus Ω rule) Let $\Omega(\succcurlyeq) = \bigcup_{\pi\in\Pi}\left\{z \in Z : z_i \sim_i \omega_{\pi(i)} \forall i\right\}$.

$$F(\succcurlyeq) = \begin{cases} IW(\succcurlyeq) \cup \Omega(\succcurlyeq) & \text{if } \succcurlyeq \in Q^n \text{ and } \succcurlyeq_1 = \cdots = \succcurlyeq_n \\ IW(\succcurlyeq) & \text{otherwise} \end{cases} \quad \text{for any } \succcurlyeq \in D.$$

This rule satisfies all the axioms except for SE. We can easily understand that F satisfies SR if we notice $z \in SR(\succcurlyeq_{E(p)})$ for any $z \in \Omega(\succcurlyeq)$.[10] Refer to the next section to see that F satisfies SN and LI. Note that $\omega \in \Omega(\succcurlyeq)$ is not always Pareto optimal. Thus SE is violated.

Example 6 (The impartial core rule) Given $\succcurlyeq$, let $Core(\succcurlyeq)$ be the set of core allocations.[11] Let F be a rule such that $z \in F(\succcurlyeq)$ if and only if there exists some $z' \in Core(\succcurlyeq)$ that is equally as just as z for any $\succcurlyeq_E \in D(\succcurlyeq)$. This rule satisfies all the axioms except for LI.

4 Another definition of rules

In the previous section, we assumed that only preferences exist primitively. The ethical observer created extended preferences from preferences, according to D.1-D.4. The rules were therefore defined on preferences, using the extended preferences as an informational basis.

In contrast, the traditional approach in the literature addressed the study of rules with the domain consisting of extended preferences.[12] In this section, we reconsider the axiomatization of the impartial Walras rule in the case of the rules being defined directly on the set of extended preferences.

A rule $f : \mathcal{D} \longrightarrow Z$ is a mapping that associates with each extended preference a nonempty subset of feasible allocations. The Walras rule $\mathcal{W}$ and the impartial Walras rule $\mathcal{IW}$ are $\mathcal{W}(\succcurlyeq_E) = W(\succcurlyeq, \omega)$ for any $\succcurlyeq_E \in \mathcal{D}$ and $\mathcal{IW}(\succcurlyeq_E) = \underset{\pi \in \Pi}{\cup} W(\succcurlyeq, \omega^\pi)$ for any $\succcurlyeq_E \in \mathcal{D}$ respectively.

A rule f satisfies $f-$**Suppes-nondiscrimination** ($f-$**SN**) if $\forall \succcurlyeq_{E'} \in D(\succcurlyeq), \forall z, z' \in Z$, z is equally as just as z', then $z \in f(\succcurlyeq_E) \Longleftrightarrow z' \in f(\succcurlyeq_E)\ \forall \succcurlyeq_E \in \mathcal{D}$.

[10] For any $z \in \Omega(\succcurlyeq)$, there exists some π such that $\min\{pq : q \sim_i z_i\} \overset{z \in \Omega(\succcurlyeq)}{\overset{\downarrow}{=}} \min\{pq : q \sim_i \omega_{\pi(i)}\} \overset{\succcurlyeq_i = \succcurlyeq_{\pi(i)}}{\overset{\downarrow}{=}} \min\{pq : q \sim_{\pi(i)} \omega_{\pi(i)}\}$. Thus $(z_i, i) \succcurlyeq_{E(p)} (\omega_{\pi(i)}, \pi(i))$.

[11] Core allocations are defined as usual by using weak and strict preference relations.

[12] Refer to d'Aspremont (1985), d'Aspremont and Gevers (2002) for example.

A rule f satisfies $f-$**Suppes-equity** ($f-$**SE**) if $z \in f(\succsim_E)$, then $\exists \succsim_{E'} \in D(\succsim)$ such that $z \in SE(\succsim_{E'})$. A rule f satisfies $f-$**Suppes-rationality** ($f-$**SR**) if $z \in f(\succsim_E)$, then $\exists \succsim_{E'} \in D(\succsim)$ such that $z \in SR(\succsim_{E'})$.

A rule f satisfies $f-$**Local Independence** ($f-$**LI**) if $\forall \succsim_E, \succsim'_E \in \mathcal{D}$, $\forall z \in Z \cap R^{nl}_{++}$ if $p(\succsim_i, z_i) = p(\succsim'_i, z_i)$ $\forall i$, then $z \in f(\succsim_E) \Longleftrightarrow z \in f(\succsim'_E)$. A rule f satisfies $f-$**Extended Local Independence** ($f-$**ELI**) if $\forall \succsim_E \in \mathcal{D}$, $\forall z \in Z \cap R^{nl}_{++}$, if $\succsim_E$ is essentially-identical to $\succsim^p_E$ around z, then $z \in f(\succsim_E) \Longleftrightarrow z \in f(\succsim^p_E)$.

The following is the counterpart of Theorem 1.

Theorem 2 *The impartial Walras rule $\mathcal{IW}$ is the unique rule satisfying $f-SN$, $f-SE$, $f-SR$ and $f-LI$.*

As in the previous section, $f-$LI is equivalent to $f-$ELI in Theorem 2. The readers may have the impression that each $f-$axiom is just a mechanical translation of the original. They would think that each axiom has a more natural form, for example, that $f-$SE should be $f(\succsim_E) \subset SE(\succsim_E)$ for any $\succsim_E \in D(\succsim)$. However, this is impossible because even if an impartial Walrasian allocation can be Suppes-equitable for *some* $\succsim_E \in D(\succsim)$, it is not necessarily Suppes-equitable for *all* $\succsim_E \in D(\succsim)$. The things are the same as other axioms. We cannot rewrite them into more natural forms.[13]

The comparison with the previous literature on the utilitarian rule makes this point clearer. It is meaningful only when comparing welfare between agents is admitted. Thus, the definition needs the set of extended utility profiles, which was the method employed in the traditional literature. By contrast, that of the impartial Walras rule only needs subjective preferences, not depending on extended preferences.

[13] Note that Suppes-equitability does not depend on configurations of initial endowments. Thus, if rewriting is possible, the second fundamental theorem of welfare economics shows that almost all Pareto otimal allocations are Suppes-equitable.

Comparing the welfare of different individuals usually makes a strong value judgment. Thus, it is risky to rely on it entirely. The approach employed in the previous section has an advantage that it carefully selects extended preferences according to D.1 to D.4. We recommend the reconsideration of the problem of interpersonal comparisons of welfare along the line of this approach.

5 Proofs

The proof of Theorem 1 follows along the same line as Nagahisa (1991). SR, SN, and SE play the same role as in individual rationality, nondiscrimination, and Pareto optimality there respectively.

Lemma 1 *The Impartial Walras rule satisfies SE, SN, and SR.*

Proof. SE: Take $z \in IW(\succcurlyeq)$ arbitrarily. Let $p \in int.\Delta^l$ be the price associated with z. Take $\succcurlyeq_{E(p)} \in D(\succcurlyeq)$. We show $z \in SE(\succcurlyeq_{E(p)})$, which completes the proof. Suppose on the contrary that there exists some $z' \in Z$ such that z' is more just than z for $\succcurlyeq_{E(p)}$. This implies that there exists some $\pi \in \Pi$ such that $(z'_i, i) \succcurlyeq_{E(p)} (z_{\pi(i)}, \pi(i))$ for all i and $(z'_i, i) \succ_{E(p)} (z_{\pi(i)}, \pi(i))$ for some i. By definition of $\succcurlyeq_{E(p)}$, this further implies $pz'_i \geq pz_{\pi(i)}$ for all i and $pz'_i > pz_{\pi(i)}$ for some i, which contradicts the feasibility of z and z'.

SN: Let $z \in IW(\succcurlyeq)$ and $p \in int.\Delta^l$ be the price associated with z. Let $z' \in Z$ be such that z is equally as just as z' for any $\succcurlyeq_E \in D(\succcurlyeq)$. By definition of IW, there is some $\pi \in \Pi$ such that for all $i \in N$,

(1) $z_i \succcurlyeq_i x_i$ for all $x_i \in R^l_+$ such that $px_i \leq p\omega_{\pi(i)}$.

Note that $\succcurlyeq_{E(p)} \in D(\succcurlyeq)$ and that z is equally as just as z' for $\succcurlyeq_{E(p)}$. We know that there is some $\rho \in \Pi$ such that

(2) $(z_{\rho(i)}, \rho(i)) \sim_{E(p)} (z'_i, i)$ for all $i \in N$.

Thus we have

(3) $pz_{\rho(i)} = \min\{pq : q \sim_{\rho(i)} z_{\rho(i)}\} = \min\{pq : q \sim_i z'_i\}$.

The first equation of (3) follows from p being the supporting price of z, and the second from the definition of $\succcurlyeq_{E(p)}$ and (2). Obviously (3) implies $pz_{\rho(i)} \leq pz'_i$ for all $i \in N$, which, due to the feasibility of z and z', implies

(4) $pz_{\rho(i)} = pz'_i$ for all $i \in N$.

Substituting (4) for (3), we know $pz'_i = \min\{pq : q \sim_i z'_i\}$ and hence

(5) p is the supporting price at z'_i.

On the other hand, (1) implies $pz_i = p\omega_{\pi(i)}$ for all $i \in N$ and hence

(6) $pz_{\rho(i)} = p\omega_{\pi(\rho(i))}$ for all $i \in N$.

(6) together with (4) implies

(7) $pz'_i = p\omega_{\pi(\rho(i))}$ for all $i \in N$,

which together with (5) assures that z' is a Walrasian allocation with agent i's endowment being agent $\pi(\rho(i))$'s endowment, which completes the proof of SN.

SR: Let $z \in IW(\succcurlyeq)$. By definition of IW, there are some $p \in int.\Delta^l$ and some $\pi \in \Pi$ such that for all $i \in N$, $z_i \succcurlyeq_i x_i$ for all $x_i \in R^l_+$ such that $px_i \leq p\omega_{\pi(i)}$. Let $x^*_{\pi(i)}$ be the best on the set $\{x_{\pi(i)} \in R^l_+ : px_{\pi(i)} \leq p\omega_{\pi(i)}\}$ with respect to $\succcurlyeq_{\pi(i)}$. Noting that $\succcurlyeq_{E(p)} \in D(\succcurlyeq)$ and p is the supporting price at z_i and $x^*_{\pi(i)}$, we have $(z_i, i) \sim_{E(p)} (x^*_{\pi(i)}, \pi(i))$. By definition of $x^*_{\pi(i)}$, we have $x^*_{\pi(i)} \succcurlyeq_{\pi(i)} \omega_{\pi(i)}$, which together with axiom of identity implies $(x^*_{\pi(i)}, \pi(i)) \succcurlyeq_{E(p)} (\omega_{\pi(i)}, \pi(i))$. Thus we conclude $(z_i, i) \succcurlyeq_{E(p)} (\omega_{\pi(i)}, \pi(i))$. This is true for all $i \in N$, and so $z \in SR(\succcurlyeq_{E(p)}, \omega) \subset \bigcup_{\succcurlyeq_E \in D(\succcurlyeq)} SR(\succcurlyeq_E, \omega)$, which completes the proof of SR. ■

Let $Z^p = \bigcup_{\pi \in \Pi} \{z \in Z : pz_i = p\omega_{\pi(i)} \ \forall i\}$.

Lemma 2 *Let F be a rule satisfying SR and SN. Then we have $F(\succcurlyeq^p) = Z^p$ for any $p \in int.\Delta^l$.*

Proof. Let $z \in F(\succsim^p)$. SR and D.1 mean that there is some $\pi \in \Pi$ such that $pz_i \geq p\omega_{\pi(i)}$ for all $i \in N$. Due to the feasibility of z, we have $pz_i = p\omega_{\pi(i)}$ for all $i \in N$, which means $z \in Z^p$. Thus we have $F(\succsim^p) \subset Z^p$.

Next, take $z \in Z^p$ arbitrarily. The previous argument ensures the existence of $z' \in F(\succsim^p) \subset Z^p$. By definition of Z^p, there exist π and ρ such that $pz_i = p\omega_{\pi(i)}$ and $pz'_i = p\omega_{\rho(i)}$ for all i. Thus $pz'_{\rho^{-1}(i)} = p\omega_i = pz_{\pi^{-1}(i)}$ for all i. By noting D.1, SN and $z' \in F(\succsim^p)$ imply $z \in F(\succsim^p)$. Thus we also have $F(\succsim^p) \supset Z^p$. ■

Lemma 3 *For any $\succsim_E \in D(\succsim)$, and any $z \in R^{nl}_{++}$, the following are equivalent to each other.*

(i) $p \in int.\Delta^l$ is the supporting price of $\succsim$ at z.

(ii) For any $(u_i)_{i\in N}$ representing $\succsim_E$, $u_i(x_i) = \alpha_i px + \beta_i + \varepsilon_i(x_i)$ for all $x_i \in R^l_+$, where $\alpha_i > 0$ and β_i are constants, and $\frac{\varepsilon_i(x_i)}{||x_i - z_i||} \longrightarrow 0$ as $x_i \longrightarrow z_i$ and $\varepsilon_i(z_i) = 0$.

Proof. (i) implies (ii): Take $(u_i)_{i\in N}$ representing $\succsim_E$arbitrarily. As u_i represents $\succsim_i$ because of the axiom of identity, the differentiability of u_i implies

$u_i(x_i) = u_i(z_i) + \sum_{h=1}^{l} \frac{\partial u_i(z_i)}{\partial x_{ih}}(x_{ih} - z_{ih}) + \varepsilon_i(x_i)$, where $\frac{\varepsilon_i(x_i)}{||x_i - z_i||} \longrightarrow 0$ as $x_i \longrightarrow z_i$ and $\varepsilon_i(z_i) = 0$

As $\frac{\partial u_i(z_i)}{\partial x_{ih}} = \alpha_i p_h$, where $\alpha_i > 0$ is the Lagrange multiplier, this equation leads us to $u_i(x_i) = \alpha_i px_i + u_i(z_i) - \alpha_i pz_i + \varepsilon_i(x_i)$. Regarding $u_i(z_i) - \alpha_i pz_i$ as β_i, we have the desired result.

(ii) implies (i): This is obvious because $\frac{\partial u_i(z_i)}{\partial x_{ih}} = \alpha_i p_h$ for all i and all h. ■

We say that F satisfies Pareto optimality (PO) if $\forall \succsim \in D$, $F(\succsim) \subset PO(\succsim)$.

Lemma 4 *For any rule F satisfying PO, it satisfies ELI if and only if it satisfies LI.*

Proof. ELI implies LI: Suppose that $p(\succsim_i, z_i) = p(\succsim'_i, z_i)$ for any i, where $\succsim, \succsim' \in D$ and any $z \in Z \cap R^{nl}_{++}$. Let $z \in F(\succsim)$. As F satisfies PO, this implies that for some $p \in int.\Delta^l$, $p(\succsim_i, z_i) = p(\succsim'_i, z_i) = p$ for any i. Lemma 3 implies that all $\succsim_E \in D(\succsim)$ and all $\succsim'_E \in D(\succsim')$ are essentially-identical to $\succsim^p_E$ around z. Thus we have $z \in F(\succsim) \overset{\overset{ELI}{\downarrow}}{\Longrightarrow} z \in F(\succsim^p) \overset{\overset{ELI}{\downarrow}}{\Longrightarrow} z \in F(\succsim')$, which completes $z \in F(\succsim) \Longrightarrow z \in F(\succsim')$. We can prove $z \in F(\succsim) \Longleftarrow z \in F(\succsim')$ similarly.

LI implies ELI: Suppose that there exists some $p \in int.\Delta^l$ such that all $\succsim_E \in D(\succsim)$ are essentially-identical to $\succsim^p_E$ around $z \in Z \cap R^{nl}_{++}$. Lemma 3 implies that p is the supporting price of $\succsim$ at z. Thus, by LI, we have $z \in F(\succsim) \Longleftrightarrow z \in F(\succsim^p)$, the desired result. ■

Proof of Theorem 1. Let F be a rule satisfying SN, SR, SE, and LI. The only remaining thing to prove is $F = IW$.

$IW \subset F$: Take $z \in IW(\succsim)$ arbitrarily. By definition of IW, i weakly prefers z_i to $\omega_{\pi(i)}$. Suppose $z \notin R^{nl}_{++}$. Then, we have $z_i \notin R^l_{++}$ for some i. If $\succsim \in Q^n$, the boundary condition requires $\omega_{\pi(i)} \succ_i z_i$, which is a contradiction. So we assume $\succsim \in IL$. Let p be such that $u_i(x) = px$ represent $\succsim_i$. As i weakly prefers z_i to $\omega_{\pi(i)}$, we have $pz_i \geq p\omega_{\pi(i)}$. As this holds for all i, we have $pz_i = p\omega_{\pi(i)}$ for all i, and hence $z \in Z^p$. Lemma 2 implies $z \in F(\succsim)$, the desired result.

Next, consider the case of $z \in R^{nl}_{++}$. Let p be an equilibrium price associated with z. We have $pz_i = p\omega_{\pi(i)}$ for all i, and hence $z \in Z^p$. Lemma 2 shows $z \in F(\succsim^p)$. As we assumed $z \in R^{nl}_{++}$, LI implies $z \in F(\succsim)$, which is the desired result.

$F \subset IW$: Take $z \in F(\succsim)$ arbitrarily. Suppose $\succsim^p \in IL$. Lemma 2 implies that there exists some $\pi \in \Pi$ such that $pz_i = p\omega_{\pi(i)}$ for all i. Thus $z \in IW(\succsim)$, the desired result.

Next, consider the case of $\succsim \in Q^n$. Assume $z \notin R^{nl}_{++}$. Then, we have $z_i \notin R^l_{++}$ for some i. SR implies that there exist some $\succsim_E \in D(\succsim)$ and some

$\pi \in \Pi$ such that $(z_i, i) \succcurlyeq_E (\omega_{\pi(i)}, \pi(i))$. As $z_i \sim_i 0$, the axiom of identity implies $(z_i, i) \sim_E (0, i)$. Thus we have

$$(\omega_{\pi(i)}, \pi(i)) \overset{\text{monotonicity}}{\underset{\downarrow}{\succ_E}} (\frac{\omega_{\pi(i)}}{2}, \pi(i)) \overset{\text{D.2}}{\underset{\downarrow}{\succcurlyeq_E}} (0, i) \sim_E (z_i, i).$$

This contradicts $(z_i, i) \succcurlyeq_E (\omega_{\pi(i)}, \pi(i))$.

Now we can assume $z \in R^{nl}_{++}$. SE implies $z \in PO(\succcurlyeq)$, and hence there exists some $p \in int.\Delta^l$ such that p is the supporting price at z. The desired result follows from the arrows below.

$$z \in F(\succcurlyeq) \overset{\text{LI}}{\Longleftrightarrow} z \in F(\succcurlyeq^p) \overset{\text{Lemma 2}}{\Longleftrightarrow} z \in IW(\succcurlyeq^p) \overset{\text{Definition of } IW}{\Longrightarrow} z \in IW(\succcurlyeq)$$

■

Proposition 1 *Let $\succcurlyeq_E$ be an extended preference satisfying two assumptions below.*

(Continuity among agents) For any $i, j \in N$, and any $x^\nu, y^\nu \in R^l_+ (\nu = 1, 2, ...,)$,

if $(x^\nu, i) \succcurlyeq_E (y^\nu, j)$ for all ν, and $x^\nu \longrightarrow x$, $y^\nu \longrightarrow y$, then $(x, i) \succcurlyeq_E (y, j)$.

(Nonsatiation among agents) For any $(x, i) \in R^l_+ \times N$, and any $j \in N$, there is some $y \in R^l_+$ such that $(x, i) \prec_E (y, j)$.

Then there exists a set of utility functions representing $\succcurlyeq_E$.

Proof. Let e be a consumption such that $e = (1, 1, ..., 1)$. Let $I = \{te \in R^l_+ : t \geq 0\}$. We define u_1 by $u_1(x) = t_x$, where $t_x e \sim_1 x$. The proof goes on with two steps.

Step 1. For any $(x, i) \in R^l_+ \times N$, $i \neq 1$, there is a unique $t \geq 0$ such that $(x, i) \sim_E (te, 1)$.

Proof of Step 1: The uniqueness is obvious from the axiom of identity and monotonicity of $\succcurlyeq_1$. We show the existence. Consider the case of $x \notin R^l_{++}$. As $(x, i) \overset{x \sim_i 0}{\underset{\downarrow}{\sim_E}} (0, i) \overset{D.2}{\underset{\downarrow}{\sim_E}} (0, 1)$, $t = 0$ is the desired one.

Next, consider the case of $x \in R^l_{++}$. The we have $(0,1) \overset{D.2}{\overset{\downarrow}{\sim}}_E (0,i) \overset{\text{monotonicity}}{\overset{\downarrow}{\prec}}_E (x,i)$. Nonsatiation among agents means that there is some $z \in R^l_+$ such that $(x,i) \prec_E (z,1)$. On the other hand there exists a unique $\widehat{te} \in I$ such that $(\widehat{te},1) \sim_E (z,1)$. We conclude that $(0,1) \prec_E (x,i) \prec_E (\widehat{te},1)$. Take a segment $[0,\widehat{te}]$. If there is no $y \in [0,\widehat{te}]$ such that $(y,1) \sim_E (x,i)$, then $[0,\widehat{te}]$ is divided into two nonempty open sets,$\{y \in [0,\widehat{te}] : (y,1) \prec_E (x,i)\}$ and $\{y \in [0,\widehat{te}] : (x,i) \prec_E (y,1)\}$, which contradicts the connectedness of $[0,\widehat{te}]$.[14] Hence we have $te \in I$ with $(te,1) \sim_E (x,i)$, which completes the proof of Step 1.

Step 2. We complete the proof.

For each $i \neq 1$, we define a continuous mapping $\varphi_{i1} : R^l_+ \longrightarrow R^l_+$ that associates each $x \in R^l_+$ with $\varphi_{i1}(x) = te$ such that $(x,i) \sim_E (te,1)$.[15]

The desired utility functions are u_1 and $u_i := u_1(\varphi_{i1})$, $(i = 2,...,n)$. The arrows below complete the proof:

$(x,i) \succsim_E (y,j) \overset{\text{def. of } \varphi_{i1} \text{ and } \varphi_{j1}}{\Longleftrightarrow} (\varphi_{i1}(x),1) \succsim_E (\varphi_{j1}(y),1) \overset{\text{the axiom of identity}}{\Longleftrightarrow}$
$(\varphi_{i1}(x),1) \succsim_1 (\varphi_{j1}(y),1)$

$\Longleftrightarrow u_1(\varphi_{i1}(x)) \geq u_1(\varphi_{j1}(y)) \overset{\text{def. of } u_i \text{ and } u_j}{\Longleftrightarrow} u_i(x) \geq u_j(y)$,

where we set $u_1 = u_1(\varphi_{11})$ and φ_{11} is identity mapping. ■

Proof of Example 5. Let us show SN. For this, it suffices to show that if $z \in \Omega(\succsim)$ and $z' \in Z$ is equally as just as z for all $\succsim_E \in D(\succsim)$, then $z' \in \Omega(\succsim)$. Take $p \in int.\Delta^l$ arbitrarily. There exist some π and ρ such that $z_i \sim_i \omega_{\pi(i)}$ and $\min\{pq : q \sim_i z_i\} = \min\{pq : q \sim_{\rho(i)} z'_{\rho(i)}\}$ for all i. Thus $\min\{pq : q \sim_{\rho(i)} z'_{\rho(i)}\} = \min\{pq : q \sim_i z_i\} = \min\{pq : q \sim_i \omega_{\pi(i)}\}$. As i and $\rho(i)$ have the same preference, this implies $z'_{\rho(i)} \sim_{\rho(i)} \omega_{\pi(i)}$, i.e., $z'_i \sim_i \omega_{\rho^{-1}(\pi(i))}$, which means $z' \in \Omega(\succsim)$.

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[14] The openness follows from continuity among agents. The nonemptyness follows from the fact that 0 belongs to the first set and $\widehat{te}$ belongs to the second set.

[15] The continuity of φ_{i1} follows from continuity among agents.

We show LI. Let $\succcurlyeq, \succcurlyeq' \in D$ and $z \in Z \cap R^{nl}_{++}$ be such that $p(\succcurlyeq_i, z_i) = p(\succcurlyeq'_i, z_i)$ for any i. Suppose $z \in F(\succcurlyeq)$. If $z \in IW(\succcurlyeq)$, $z \in IW(\succcurlyeq') \subset F(\succcurlyeq')$, which completes the proof. Thus, we consider the case of $z \in \Omega(\succcurlyeq)$, where $\succcurlyeq \in Q^n$ and $\succcurlyeq_1 = \cdots = \succcurlyeq_n$. As everyone has the identical preference, we can suppose that $p = p(\succcurlyeq_i, z_i) = p(\succcurlyeq'_i, z_i)$ for any i. The definition of $\Omega(\succcurlyeq)$ and the feasibility of z imply that there exists some $\pi \in \Pi$ such that $pz_i = p\omega_{\pi(i)} \forall i$, which implies $z \in IW(\succcurlyeq)$, and hence $z \in IW(\succcurlyeq') \subset F(\succcurlyeq')$, the desired result. ■

We prove Theorem 2 while using Theorem 1 as a lemma.

Lemma 5 *Let f be a rule satisfying $f-$SR and $f-$SN. Then, for any $\succcurlyeq^p \in IL$,*

$f(\succcurlyeq^p_E) = Z^p$ for any $\succcurlyeq_E \in \mathcal{D}$.

Proof. The proof follows along the same line as Lemma 2, so it is omitted. ■

A rule f satisfies $f-$Pareto optimality ($f-$PO) if $\forall \succcurlyeq_E \in \mathcal{D}$, $f(\succcurlyeq_E) \subset PO(\succcurlyeq)$.

Lemma 6 *For any f satisfying $f-$PO, it satisfies $f-$ELI if and only if it satisfies $f-$ LI.*

Proof. The proof is in parallel with Lemma 4, so it is omitted. ■

Lemma 7 *For any f satisfying $f-$SE, $f-$SR, $f-$SN, and $f-$LI, $f(\succcurlyeq_E) = f(\succcurlyeq_{E'})$ for any $\succcurlyeq_E, \succcurlyeq_{E'} \in D(\succcurlyeq)$ and any $\succcurlyeq \in D$.*

Proof. Lemma 2 completes the proof of the case of $\succcurlyeq \in IL$.

We consider the case of $\succcurlyeq \in Q^n$. Take $z \in f(\succcurlyeq_E)$ arbitrarily. By using $f-$SR and $f-$SE, the same argument as in the proof of Theorem 1 leads us to $z \in PO(\succcurlyeq) \cap R^{nl}_{++}$. By letting $p \in int.\Delta^l$ be the supporting price at z, Lemma 3 shows that $\succcurlyeq_{E(p)}$is essentially-identical to $\succcurlyeq^p_E$ around z. As $\succcurlyeq_{E(p)}, \succcurlyeq_E \in D(\succcurlyeq)$ and $z \in f(\succcurlyeq_E)$, this together with $f-$ELI implies $z \in f(\succcurlyeq^p_E)$. Now that we have $\succsim_{E(p)}, \succsim_{E'} \in D(\succsim)$ and $z \in f(\succsim_E^p)$, $f-$ELI is applied again, and hence we conclude $z \in f(\succsim_{E'})$, the desired result. ■

Proof of Theorem 2. First, we show that $\mathcal{IW}$ satisfies all the axioms. The clue is that $\mathcal{IW}(\succsim_E) = IW(\succsim)$ for any $\succsim_E \in D(\succsim)$ and any $\succsim \in D$. Thus $\mathcal{IW}$ satisfies $f-$SE because IW satisfies SE. We can show the satisfaction of the other axioms in the same way.

Next, we show the uniqueness. Let f be a rule satisfying all the axioms. Let $F : D \longrightarrow Z$ be a rule such that

$$F(\succsim) = \bigcup_{\succsim_E \in D(\succsim)} f(\succsim_E).$$

It is easy to see that F satisfies SN, SE, SR, and LI. Thus Theorem 1 shows that $F = IW$, which together with Lemma 7 says that $IW(\succsim) = f(\succsim_E)$ for any $\succsim_E \in D(\succsim)$. We conclude $f = \mathcal{IW}$. ■

6 Conclusion

In this paper, we combined two studies advanced independently in social choice, the studies of interpersonal comparisons of welfare and that of axiomatic analysis of resource allocation problems.[16] We conclude with one more remark below.

It seems to have not very much been studied so far to what extent it is possible to create extended preferences from a given preference profile.[17] Most of the literature on interpersonal comparisons of welfare assumes extended preferences a priori and does not ask for the ground.[18] In contrast, we have stated that there must be a convincing basis for the extended preferences created from

[16] Chamber and Hayashi (2017) proposed an alternative axiomatic approach of the Walras rule that can cope with income distribution problems.

[17] Refer to Hammond (1989), which is a comprehensive survey of this topic, for more details.

[18] Refer to d'Aspremont (1985) for example, which defines an extended utility profile U, a real-valued function defined on $X \times N$, where X is the set of alternatives, finite or infinite, unstructured. A social welfare functional, a generalization of Arrow's social welfare function, is defined on $\mathcal{U}$, the set of *all logically possible* extended utility profiles. We can see the same setting in almost every literature from Sen (1970) to Yamamura (2017).

a given preference profile. D.1-D.4 is the basis. We hope that this paper will provide new insights into research in this direction.

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