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Strategy-Proof Mechanism Design with Boundedly Rational Agents *

Katsuhiko Nishizaki †‡

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Abstract

This paper considers an implementation problem with bounded rationality of the agents. Bounded rationality presented here means that the agent might choose the agent's best response which is different from the agent's dominant strategy. To describe such a behavior, this paper introduces a new notion of equilibrium, called $(n-k)$-dominant strategy Nash equilibrium, in which at most $k \in \{0,1,\ldots,n\}$ boundedly rational agents might choose their best responses which are different from their dominant strategies and at least $(n-k)$ rational agents choose their dominant strategies. In addition, to show what a socially optimal outcome collectively chosen under the existence of boundedly rational agents, this paper introduces a new notion of implementation, called k-secure implementation, which is double implementation in dominant strategy equilibria and $(n-k)$-dominant strategy Nash equilibria. In specific environments, majority rule satisfies k-secure implementability, but not secure implementability (Saijo, T., T. Sjöström, and T. Yamato (2007) "Secure Implementation," *Theoretical Economics* 2, pp.203-229) which is equivalent to n-secure implementability.

Keywords: Strategy-Proofness, Bounded Rationality, Dominant Strategy Implementation, Secure Implementation, Binary Voting.

JEL Classification: C72, C92, D71, D72, D78.

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1 Introduction

In social choice theory, **strategy-proofness** is a standard condition for strategic non-manipulability. [1] This condition requires that truthful revelation is a dominant strategy in the direct revelation mechanism associated with a social choice function satisfying strategy-proofness (strategy-proof mechanism). A social choice function is a function which associates each agents' preferences with an outcome. The direct revelation mechanism associated with a social choice function is a mechanism such that a strategy is revealing a preference and an outcome is assigned by the associated social choice function depending on revealed preferences. Strategy-proofness is also a standard condition in implementation theory because it is necessary for **dominant strategy implementation**, known as the revelation principle for dominant strategy implementation (Dasgupta, Hammond, and Maskin, 1979). [2] Dominant strategy implementation requires that the set of outcomes achieved by dominant strategy equilibria coincides with the set of outcomes assigned by the social choice function. Mizukami and Wakayama (2007) and Saijo, Sjöström, and Yamato (2007) independently showed that the social choice function is dominant strategy implementable by the direct revelation mechanism associated with it if and only if it satisfies strategy-proofness and **weak non-bossiness** (Mizukami and Wakayama, 2007; Saijo, Sjöström, and Yamato, 2007).

Dominant strategy implementation is attractive because dominant strategy equilibrium does not require any strategic coordination among the agents to achieve itself. This attraction depends on the assumption that each agent chooses the agent's dominant strategy to induce the agent's most preferred outcome if there is a dominant strategy equilibrium. Although this assumption is reasonable when the indifference in preferences is not allowed for each agent, it is questionable whether this assumption is reasonable when the indifference in preferences is allowed for some agent because there might be the agent's best response which is different from the agent's dominant strategy. [3] This assumption is a sufficient condition for guaranteeing that each agent seeks the agent's most preferred outcome, but not a necessary condition, because the agent might be able to induce such an outcome by choosing the agent's best response which is different from the agent's dominant strategy.

Several experimental results also questioned whether the above assumption about the agents' behaviors is reasonable. They showed that there are several subjects who do not reveal their true preferences in strategy-proof mechanisms. [4] On the other hands, they showed that some of the subjects reveal their preferences which are their best responses to other agents' revelations. On the basis of such experimental results in addition to the theoretical question about the agents' behaviors, this paper reconsiders strategy-proof mechanisms in terms of the assumption about bounded rationality of the agents, that is, this paper

[1] See Barberà (2011), Bossert and Weymark (2008), and Salles (2014) for basic ideas and recent developments of social choice theory.

[2] See Baliga and Sjöström (2008), Corchón (2007), Jackson (2014), Maskin (2008) for basic ideas and recent developments of implementation theory.

[3] Note that strategy-proofness requires that truthful revelation is a dominant strategy for each agent in the direct revelation mechanism associated with a social choice function satisfying strategy-proofness, but not preclude the existence of other best responses including dominant strategies which are different from truthful revelation. See Mizukami and Wakayama (2007) and Saijo, Sjöström, and Yamato (2007) for multiple equilibrium problems with dominant strategy implementation.

[4] See Attiyeh, Franciosi, and Isaac (2000), Cason, Saijo, Sjöström, and Yamato (2006), and Kawagoe and Mori (2001) for experiments on the pivotal mechanism (Clarke, 1971). In addition, see Kagel, Harstad, and Levin (1987) and Kagel and Levin (1993) for experiments on second-price auction.

considers the possibility that some agent chooses the agent's best response which is different from the agent's truthful revelation, like satisficing (Simon, 1947, 1956), in construction of strategy-proof mechanisms. This paper defines such an agent as a **boundedly rational agent** in contrast to a **rational agent** who reveal the agent's true preference in the strategy-proof mechanism.

This paper closely relates to those of Saijo, Sjöström, and Yamato (2007) who considered a implementation problem with the rationality of the agents on the basis of the above experimental results. Many strategy-proof mechanisms have multiple Nash equilibria when the indifference in preferences is allowed. Such equilibria might achieve a socially non-optimal outcome in contrast to truthful revelations which achieve a socially optimal outcome. [5] Saijo, Sjöström, and Yamato (2007) pointed out the existence of such Nash equilibria as the cause for the ineffectiveness of strategy-proof mechanisms in experiments and introduced **secure implementation** which is double implementation (Maskin, 1985) in dominant strategy equilibria and Nash equilibria. In addition, Saijo, Sjöström, and Yamato (2007) showed that the social choice function is securely implementable if and only if it satisfies strategy-proofness and the **rectangular property** (Saijo, Sjöström, and Yamato, 2007). [6] On the basis of this characterization, several researchers have studied the possibility of secure implementation in the following environments: voting environments (Saijo, Sjöström, and Yamato, 2007; Berga and Moreno, 2009), public good economies (Saijo, Sjöström, and Yamato, 2007; Nishizaki, 2013, 2018a), pure exchange economies (Nishizaki, 2014), production economies (Saijo, Sjöström, and Yamato, 2007; Kumar, 2013; Nishizaki 2018b), allotment economies (Bochet and Sakai, 2009), economies with indivisible private goods and monetary transfer (Fujinaka and Wakayama, 2008), queueing problems (Nishizaki, 2012), Shapley-Scarf housing markets (Fujinaka and Wakayama, 2011), and school choice (Mizukami and Wakayama, 2017). These studies mostly indicated that secure implementability is so strong and there is rarely a non-trivial securely implementable social choice function. [7]

In response to bounded rationality of the agents and the difficulty of secure implementation, several researchers might think that dominant strategy implementation should be replaced by other notion such as Nash implementation. However, other notion of implementation also has a problem with strategic coordination among the agents to establish an associated equilibrium. Together with this coordination problem in mind, this paper introduces an intermediate notion between dominant strategy implementation and secure implementation in terms of the rationality of the agents, called ***k*-secure implementation**. This notion is double implementation in dominant strategy equilibria and **$(n-k)$-dominant strategy Nash equilibria** which is introduced in this paper in terms of the rationality of the agents. In an environment with $n \geq 2$ agents, a $(n-k)$-dominant strategy Nash equilibrium is a Nash equilibrium in which at most $k \in \{0, 1, \ldots, n\}$ boundedly rational agents might choose their best responses which are different from

[5]Repullo (1985) showed that if the social choice function is dominant strategy implementable by an indirect mechanism, but not the direct revelation mechanism associated with it, then the indirect mechanism does not implement it in Nash equilibria. This means that the mechanism which implements the social choice function in dominant strategy equilibria might have Nash equilibria which do not achieve a socially optimal outcome. See Saijo, Sjöström, and Yamato (2003) for the example.

[6]See Mizukami and Wakayama (2017) for an alternative characterization of secure implementability in terms of restricted monotonicity (Mizukami and Wakayama, 2017) which is stronger than Maskin monotonicity (Maskin, 1977, 1999) by definition. In environments with partially honest agents, Saporiti (2014) showed that strategy-proofness is a necessary and sufficient condition for secure implementability.

[7]The exception was shown by Saijo, Sjöström, and Yamato (2007) in public good economies, Nishizaki (2014) in pure exchange economies, Saijo, Sjöström, and Yamato (2007), Kumar (2013), and Nishizaki (2018b) in production economies.

their dominant strategies and at least $(n-k)$ rational agents choose their dominant strategies. This notion is equivalent to dominant strategy equilibrium if $k=0$, that is, all the agents are rational and choose their dominant strategies. This implies that k-secure implementation is equivalent to dominant strategy implementation if $k=0$. On the other hand, this notion is equivalent to Nash equilibrium if $k=n$, that is, similar to Saijo, Sjöström, and Yamato (2007), all the agents might be boundedly rational and choose their best responses which are different from their dominant strategies. This implies that k-secure implementation is equivalent to secure implementation if $k=n$. If $1 \leq k \leq n-1$, then k-dominant strategy Nash equilibrium is an intermediate notion between dominant strategy equilibrium and Nash equilibrium and k-secure implementation is also an intermediate notion between dominant strategy implementation and secure implementation. k-secure implementation is attractive theoretically and practically for the following reasons: (i) if $k \leq n-1$, then this notion is in general weaker than secure implementation and there might be a non-trivial k-securely implementable social choice function by the associated direct revelation mechanism, (ii) this notion solves coordination problems because dominant strategy equilibrium plays a role of focal point (Schelling, 1960), and (iii) this notion reflects bounded rationality of the actual agents in some degree because several subjects choose their best responses which are different from their truthful revelations in experiments.

In general environments, this paper shows that the social choice function is k-securely implementable by the direct revelation mechanism associated with it if and only if it satisfies strategy-proofness and the ***k*-rectangular property** which is introduced in this paper as an intermediate notion between weak non-bossiness which is a necessary condition for dominant strategy implementability by the associated direct revelation mechanism and the rectangular property which is a necessary condition for secure implementability. In addition, this paper investigates k-secure implementability by the associated direct revelation mechanism in binary voting environments and shows that majority rule is k-securely implementable, but not securely implementable, when the number of boundedly rational agents k is less than $n^a(u)-(n/2)$, where $n^a(u)$ is the number of agents who prefer a to another outcome at the profile of utility functions for all the agents u.

This paper also closely relates to those of Eliaz (2002) who considered a implementation problem with the rationality of the agents in term of the number of irrational agents. Eliaz (2002) considered the possibility that some agent, called a faulty agent, chooses the agent's any strategy in contrast to a boundedly rational agent presented here, who might choose the agent's best response which is different from the agent's dominant strategy. On the basis of this possibility, Eliaz (2002) introduced ***k*-fault tolerant implementation** and characterized k-fault tolerantly implementable social choice correspondences. [8] This notion is defined as the implementation in ***k*-fault tolerant Nash equilibria** which are an intermediate notion between dominant strategy equilibria and Nash equilibria. In addition, k-fault tolerant implementation requires that any deviation from the k-fault tolerant Nash equilibrium by faulty agents achieves a socially optimal outcome. k-fault tolerant implementation is equivalent to Nash implementation if $k=0$ and in general stronger than dominant strategy implementation if $k=n-1$ because $(n-1)$-fault tolerant

[8] In addition, Eliaz (2002) showed a mechanism which k-fault tolerantly implements the constrained Walrasian allocation in pure exchange economies with $n \geq 2$ agents when the number of faulty agents lie between 0 and $(1/2)n-1$. Mandler (2014) also studied the faultiness of the agents in pure exchange economies. Mandler (2014) introduced an efficiency condition independent of the number of faulty agents and showed a mechanism which implements such efficient allocations in a variant of Nash equilibria.

implementation requires dominant strategy implementation and that any deviation from the dominant strategy equilibrium by at most $(n-1)$ faulty agents also achieves a socially optimal outcome. This implies that $(n-1)$-fault tolerant implementation is also in general stronger than k-secure implementation for each $k \in \{0, 1, \ldots, n-1\}$ by definition because such deviations include $(n-k)$-dominant strategy Nash equilibria for each $k \leq n-1$.

The remainder of this paper is organized as the following six sections. Section 2 introduces the basic notation and definitions. In addition, the notion of equilibrium is introduced in this section. Especially, the rationality of the agents and the relationship between $(n-k)$-dominant strategy Nash equilibrium and k-fault tolerant Nash equilibrium are discussed in Subsection 2.3. Section 3 introduces the notion of implementation including k-secure implementation and k-fault tolerant implementation. Section 4 shows a characterization of k-securely implementable social choice functions by the associated direct revelation mechanism respectively in general environments. In addition, Section 5 shows specific environments in which majority rule is k-securely implementable, but not securely implementable. The results on the relationship between the k-rectangular property and strong non-bossiness and the relationship between the k-rectangular property and the rectangular property are presented in Appendix.

2 Model

The model presented here mostly follows those of Saijo, Sjöström, and Yamato (2007). The major difference between the two models depends on their assumption about the rationality of the agents. By introducing secure implementation, Saijo, Sjöström, and Yamato (2007) implicitly assumed that all of the agents might be boundedly rational and choose their best responses which are different from their dominant strategies even though they can choose their dominant strategies. On the other hand, by introducing $(n-k)$-dominant strategy Nash equilibrium in Subsection 2.3, this paper explicitly assumes that some of the agents might be boundedly rational and choose such best responses although other rational agents precisely choose their dominant strategies. [9]

2.1 Basic Notation and Definitions

For each sets X, Y, "$X \subseteq Y$" means that "X is a subset of Y" and "$X \subset Y$" means that "X is a proper subset of Y". For each set X, "$|X|$" means "the cardinality of X".

There are $n \geq 2$ agents who collectively choose an outcome. Let $I \equiv \{1, \ldots, n\}$ be the set of the **agents** and A be the set of the **outcomes**. Each agent has a preference for the outcomes, which is represented by an utility function defined on A. For each $i \in I$, let $u_i \colon A \to \mathbb{R}$ be an **utility function for agent *i*** and U_i be the set of utility functions for agent i. For each $N \subseteq I$, let $u_N \equiv (u_i)_{i \in N}$ be a profile of utility functions for the agents in the set of agents N and $U_N \equiv \prod_{i \in N} U_i$ be the set of profiles of utility functions for the agents in the set of agents N. In addition, let $u \equiv u_I$ and $U \equiv U_I$.

A socially optimal outcome is characterized by a social choice function defined on U. Let $f \colon U \to A$ be a **social choice function**. For each $u \in U$, let $f(u) \in A$ be the socially optimal outcome assigned by

[9] Introducing a notion of equilibrium is a way of describing a behavioral principle of the agents. Other way of describing such a principle is introducing a choice function for an agent, which assigns an outcome to each element in the power set of outcomes. See de Clippel (2014) and Korpela (2012) for implementation theory based on choice functions.

the social choice function f at the profile of utility functions for all the agents u.

An outcome is collectively chosen by all the agents through a mechanism. Let $\Gamma \equiv (S, g)$ be a **mechanism** defined as follows.

(i) For each $i \in I$, let s_i be a **strategy for agent *i* under the mechanism *Γ*** and S_i be the set of strategies for agent i under the mechanism Γ. For each $N \subseteq I$, let $s_N \equiv (s_i)_{i \in N}$ be a profile of strategies for the agents in the set of agents N under the mechanism Γ and $S_N \equiv \prod_{i \in N} S_i$ be the set of profiles of strategies for the agents in the set of agents N under the mechanism Γ. In addition, let $s \equiv s_I$ and $S \equiv S_I$.

(ii) Let $g \colon S \to A$ be the **outcome function under the mechanism *Γ***. For each $s \in S$, let $g(s) \in A$ be the outcome assigned by the mechanism Γ at the profile of strategies for all the agents s.

Let $\Gamma^f \equiv (U, g^f)$ be the **direct revelation mechanism associated with a social choice function *f*** such that $g^f = f$.

2.2 Basic Notion of Equilibrium

This paper focuses on bounded rationality of the agents under a mechanism with complete information. Each rational agent is assumed to choose the agent's best response, especially the agent's dominant strategy when feasible. On the other hand, each boundedly rational agent is assumed to choose the agent's best response which might be different from the agent's dominant strategy even though the agent can choose it. To describe such an assumption in implementation problems, this subsection introduces dominant strategy equilibrium and Nash equilibrium.

For each $u \in U$ and each $i \in I$, the strategy $s_i \in S_i$ is a **dominant strategy for agent *i* in the mechanism *Γ* at the profile of utility functions *u*** if and only if for each $s'_i \in S_i$ and each $s'_{I \setminus \{i\}} \in S_{I \setminus \{i\}}$,

$$u_i(g(s_i, s'_{I \setminus \{i\}})) \geq u_i(g(s'_i, s'_{I \setminus \{i\}})).$$

For each $u \in U$ and each $i \in I$, let $DS_i(\Gamma, u) \subseteq S_i$ be the set of dominant strategies for agent i in the mechanism Γ at the profile of utility functions u.

Definition 1. For each $u \in U$, the profile of strategies $s \in S$ is a **dominant strategy equilibrium in the mechanism *Γ* at the profile of utility functions *u*** if and only if

$$s_i \in DS_i(\Gamma, u) \text{ for each } i \in I.$$

For each $u \in U$, let

$$DSE(\Gamma, u) \equiv \prod_{i \in I} DS_i(\Gamma, u)$$

be the set of dominant strategy equilibria in the mechanism Γ at the profile of utility functions u.

For each $u \in U$, each $i \in I$, and each $s_{I \setminus \{i\}} \in S_{I \setminus \{i\}}$, the strategy $s_i \in S_i$ is a **best response for agent *i* to the profile of strategies for other agents $s_{I \setminus \{i\}}$ in the mechanism *Γ* at the profile of utility functions *u*** if and only if for each $s'_i \in S_i$,

$$u_i(g(s_i, s_{I \setminus \{i\}})) \geq u_i(g(s'_i, s_{I \setminus \{i\}})).$$

For each $u \in U$, each $i \in I$, and each $s_{I\setminus\{i\}} \in S_{I\setminus\{i\}}$, let $BR_i(\Gamma, u; s_{I\setminus\{i\}}) \subseteq S_i$ be the set of best responses for agent i to the profile of strategies for other agents $s_{I\setminus\{i\}}$ in the mechanism Γ at the profile of utility functions u.

Definition 2. For each $u \in U$, the profile of strategies $s \in S$ is a **Nash equilibrium in the mechanism Γ at the profile of utility functions u** if and only if

$$s_i \in BR_i(\Gamma, u; s_{I\setminus\{i\}}) \text{ for each } i \in I.$$

For each $u \in U$, let

$$NE(\Gamma, u) \equiv \prod_{i \in I} BR_i(\Gamma, u; s_{I\setminus\{i\}})$$

be the set of Nash equilibria in the mechanism Γ at the profile of utility functions u.

2.3 Bounded Rationality of Agents

Several researchers have described boundedly rational agents from various perspectives. Especially, Eliaz (2002) focused on the number of irrational agents, similar to this paper. This subsection introduces the main notion of equilibrium presented here and discusses the relationship between it and the notion of equilibrium introduced by Eliaz (2002).

2.3.1 $(n-k)$-Dominant Strategy Nash Equilibrium

Under the mechanism in which there is a dominant strategy equilibrium at the profile of utility functions, implementation theory mostly assumes that all the agents are rational and they choose their dominant strategies. Although this assumption is reasonable when the indifference in preferences is not allowed for each agent, it is questionable whether this assumption is reasonable when the indifference in preferences is allowed for some agent because there might be the agent's best response which is different from the agent's dominant strategy. In addition, it is observed that several subjects does not choose their dominant strategies even though they can choose them in several experiments. Alternatively, some of them choose their best responses which are different from their dominant strategies. On the basis of such theoretical question and experimental results, this paper assumes that there might be several boundedly rational agents who choose their best responses which are different from their dominant strategies. This assumption is described as the following notion of equilibrium, called **$(n-k)$-dominant strategy Nash equilibrium**, defined as an intermediate notion between dominant strategy equilibrium and Nash equilibrium.

This paper assumes that there are at most $k \in \{0, 1, \ldots, n\}$ boundedly rational agents. In addition, it is assumed that the boundedly rational agents might choose their best responses which are different from their dominant strategies.

Definition 3. For each $k \in \{0, 1, \ldots, n\}$ and each $u \in U$, the profile of strategies $s \in S$ is a **$(n-k)$-dominant strategy Nash equilibrium in the mechanism Γ at the profile of utility functions u** if and only if there is $K \subseteq I$ with $|K| \leq k$ such that

$$s_i \in BR_i(\Gamma, u; s_{I\setminus\{i\}}) \text{ for each } i \in K \text{ and } s_j \in DS_j(\Gamma, u) \text{ for each } j \in I \setminus K.$$

In Definition 3, K is the set of boundedly rational agents and $I \setminus K$ is the set of rational agents. At the $(n-k)$-dominant strategy Nash equilibrium, although at least $(n-k)$ rational agents in $I \setminus K$ choose their dominant strategies, at most k boundedly rational agents in K might choose their best responses which are different from their dominant strategies.

By definition, $(n-k)$-dominant strategy Nash equilibrium is equivalent to dominant strategy equilibrium if $k = 0$ and Nash equilibrium if $k = n$. If $1 \leq k \leq n-1$, then $(n-k)$-dominant strategy Nash equilibrium is an intermediate notion between Nash equilibrium and dominant strategy equilibrium. In addition, $(n-k)$-dominant strategy Nash equilibrium is in general stronger than $(n-k')$-dominant strategy Nash equilibrium if $k < k'$ by definition. In fact, Example 1 shows these relationships.

Example 1. We consider the following three-person game: $I \equiv \{1,2,3\}$, $S_1 \equiv \{U,D\}$, $S_2 \equiv \{L,R\}$, $S_3 \equiv \{A,B\}$, and the payoffs are given as follows.

A	L	R
U	2, 2, 2	1, 2, 1
D	2, 1, 1	1, 1, 1

B	L	R
U	2, 2, 1	1, 1, 1
D	1, 1, 1	1, 1, 1

where the i-th coordinate of each profile of payoffs means agent i's payoff. This game has the following five Nash equilibria: (U, L, A), (D, L, A), (U, R, A), (D, R, A), and (D, R, B). Especially, (U, L, A) is the unique (weakly) dominant strategy equilibrium. In this game, we find that

(i) (U, L, A), (D, L, A), (U, R, A), (D, R, A), and (D, R, B) are 0-dominant strategy Nash equilibria,

(ii) (U, L, A), (D, L, A), (U, R, A), and (D, R, A) are 1-dominant strategy Nash equilibria,

(iii) (U, L, A), (D, L, A), and (U, R, A) are 2-dominant strategy Nash equilibria, and

(iv) (U, L, A) is the unique 3-dominant strategy Nash equilibria.

For each $k \in \{0,1,\ldots,n\}$ and each $u \in U$, let

$$(n-k)DSNE(\Gamma,u) \equiv \bigcup_{K \subseteq I:\ |K| \leq k} \left\{ \prod_{i \in K} BR_i(\Gamma,u;s_{I\setminus\{i\}}) \times \prod_{j \in I \setminus K} DS_j(\Gamma,u) \right\}$$

be the set of $(n-k)$-dominant strategy Nash equilibria in the mechanism Γ at the profile of utility functions u. Note that $nDSNE(\Gamma,u) = DSE(\Gamma,u)$ and $0DSNE(\Gamma,u) = NE(\Gamma,u)$ for each $u \in U$.

2.3.2 Relationship between $(n-k)$-Dominant Strategy Nash Equilibrium and k-Fault Tolerant Nash Equilibrium

In implementation problems, Eliaz (2002) described irrational agents, called faulty agents, as the agents who are more irrational than boundedly rational agents presented here. Each faulty agent is assumed to behave in any possible manner, that is, the faulty agent might not even choose the agent's best response. To counteract the influence exerted by such a behavior, Eliaz (2002) introduced ***k*-fault tolerant Nash equilibrium**.

Eliaz (2002) assumed that each rational agent considers that there are at most $k \in \{0,1,\ldots,n-1\}$ faulty agents other than the agent. [10] In addition, it is assumed that at most k faulty agents choose their

[10] Eliaz (2002) called a rational agent by a non-faulty agent.

any strategies respectively. For each $u \in U$, each $i \in I$, each $k \in \{0,1,\ldots,n-1\}$, each $K \subseteq I$ with $|K| \leq k$, and each $s_{I \setminus \{K \cup \{i\}\}} \in S_{I \setminus \{K \cup \{i\}\}}$, the strategy $s_i \in S_i$ is a ***k*-fault tolerant best response for agent *i* to the profile of strategies for other rational agents $s_{I \setminus \{K \cup \{i\}\}}$ in the mechanism *Γ* at the profile of utility functions *u*** if and only if for each $s'_i \in S_i$ and each $s'_K \in S_K$,

$$u_i(g(s_i, s_{I \setminus \{K \cup \{i\}\}}, s'_K)) \geq u_i(g(s'_i, s_{I \setminus \{K \cup \{i\}\}}, s'_K)).$$

In the definition of k-fault tolerant best response, K is the set of faulty agents and $I \setminus \{K \cup \{i\}\}$ is the set of rational agents other than agent $i \in I$. k-fault tolerant best response requires that agent i has no incentive to deviate from the strategy given the strategies of other "rational" agents in $I \setminus \{K \cup \{i\}\}$ in contrast to best response for agent i to the profile of strategies for "all" other agents. By definition, k-fault tolerant best response is equivalent to best response if $k = 0$ and dominant strategy if $k = n-1$. If $1 \leq k \leq n-2$, then k-fault tolerant best response is an intermediate notion between best response and dominant strategy. In addition, k-fault tolerant best response is in general weaker than k'-fault tolerant best response if $k < k'$ by definition. For each $u \in U$, each $i \in I$, each $k \in \{0,1,\ldots,n-1\}$, each $K \subseteq I$ with $|K| \leq k$, and each $s_{I \setminus \{K \cup \{i\}\}} \in S_{I \setminus \{K \cup \{i\}\}}$, let $kFTBR_i(\Gamma, u; s_{I \setminus \{K \cup \{i\}\}}) \subseteq S_i$ be the set of k-fault tolerant best responses for agent i to the profile of strategies for other rational agents $s_{I \setminus \{K \cup \{i\}\}}$ in the mechanism Γ at the profile of utility functions u.

Definition 4. For each $k \in \{0,1,\ldots,n-1\}$ and each $u \in U$, the profile of strategies $s \in S$ is a ***k*-fault tolerant Nash equilibrium in the mechanism *Γ* at the profile of utility functions *u*** if and only if for each $K \subseteq I$ with $|K| \leq k$,

$$s_i \in kFTBR_i(\Gamma, u; s_{I \setminus \{K \cup \{i\}\}}) \text{ for each } i \in I.$$

In Definition 4, K is the set of faulty agents and $I \setminus K$ is the set of rational agents. At the k-fault tolerant Nash equilibrium, although at least $(n-k)$ rational agents in $I \setminus K$ choose their best responses mutually, at most k faulty agents in K choose their any possible strategies.

Remark 1. At the $(n-k)$-dominant strategy Nash equilibrium, each boundedly rational agent is assumed to choose the agent's equilibrium strategy. On the other hand, at the k-fault tolerant Nash equilibrium, each faulty agent is not necessarily assumed to choose the agent's equilibrium strategy. k-fault tolerant Nash equilibrium is a robust notion of equilibrium in the sense that each rational agent has no incentive to deviate from the agent's equilibrium strategy even if faulty agents choose their any strategies. [11]

By definition, k-fault tolerant Nash equilibrium is equivalent to Nash equilibrium if $k = 0$ and dominant strategy equilibrium if $k = n-1$. If $1 \leq k \leq n-2$, then k-fault tolerant Nash equilibrium is an intermediate notion between Nash equilibrium and dominant strategy equilibrium. In addition, k-fault tolerant Nash equilibrium is in general weaker than k'-fault tolerant Nash equilibrium if $k < k'$ by definition. In fact, Example 2 shows these relationships.

[11] At the dominant strategy equilibrium, all the agents choose their dominant strategy. If a faulty agent chooses the agent's any strategy, then each rational agent has no incentive to deviate from the dominant strategy because the incentive to choose it does not depend on other agents' strategies. This implies that dominant strategy equilibrium is so robust to such faulty behaviors. On the other hand, at the Nash equilibrium, all the agents choose their best responses to other agents' strategies. If a faulty agent chooses the agent's any strategy, then each rational agent might have an incentive to deviate from the best response because the incentive to choose it depends on other agents' strategies. This implies that Nash equilibrium is not robust to such faulty behaviors.

Example 2. Similar to Example 1, we consider the following three-person game: $I \equiv \{1,2,3\}$, $S_1 \equiv \{U,D\}$, $S_2 \equiv \{L,R\}$, $S_3 \equiv \{A,B\}$, and the payoffs are given as follows.

A	L	R
U	2, 2, 2	1, 2, 1
D	2, 1, 1	1, 1, 1

B	L	R
U	2, 2, 1	1, 1, 1
D	1, 1, 1	1, 1, 1

As stated in Example 1, this game has the following five Nash equilibria: (U, L, A), (D, L, A), (U, R, A), (D, R, A), and (D, R, B). Especially, (U, L, A) is the unique (weakly) dominant strategy equilibrium. In this game, we find that

(i) (U, L, A), (D, L, A), (U, R, A), (D, R, A), and (D, R, B) are 0-fault tolerant Nash equilibria,

(ii) (U, L, A) and (D, R, A) are 1-fault tolerant Nash equilibria, and

(iii) (U, L, A) is the unique 2-fault tolerant Nash equilibrium.

At the 0-fault tolerant Nash equilibria, each rational agent considers that there is no faulty agent other than the agent. Therefore, each rational agent has no incentive to deviate from all the equilibria.

At the 1-fault tolerant Nash equilibria, each rational agent considers that there is at most one faulty agent other than the agent. At (U, L, A), each rational agent has no incentive to deviate from the agent's strategy even if one other agent is faulty because the strategy is the agent's (weakly) dominant strategy. At (D, R, A), if agents 1 and 2 are rational, then they have no incentive to deviate from their strategies respectively even if agent 3 is faulty and choose any possible strategy. In addition, rational agents 1 and 3 have no such an incentive even if agent 2 is faulty and rational agents 2 and 3 do even if agent 1 is faulty. (D, L, A) is not a 1-fault tolerant Nash equilibrium because agent 1 has an incentive to deviate from (D, L, B) by choosing U when agent 3 is faulty and deviates from (D, L, A) by choosing B. Similarly, (U, R, A) is not a 1-fault tolerant Nash equilibrium because agent 2 has an incentive to deviate from (U, R, B) by choosing L when agent 3 is faulty and deviates from (U, R, A) by choosing B and (D, R, B) is not because agent 1 has an incentive to deviate from (D, L, B) by choosing U when agent 2 is faulty and deviates from (D, R, B) by choosing L.

At the 2-fault tolerant Nash equilibria, each rational agent considers that there are at most two faulty agents other than the agent. At (U, L, A), each rational agent has no incentive to deviate from the agent's strategy even if two other agents are faulty because the strategy is the agent's (weakly) dominant strategy.

Remark 2. By definition, 0-fault tolerant Nash equilibrium is equivalent to 0-dominant strategy Nash equilibrium because these are equivalent to Nash equilibrium. On the other hand, $(n-1)$-fault tolerant Nash equilibrium is equivalent to n-dominant strategy Nash equilibrium by definition because these are equivalent to dominant strategy equilibrium.

Remark 3. By Examples 1 and 2, we find that (D, L, A) and (U, R, A) are 2-dominant strategy Nash equilibria, but not 1-fault tolerant Nash equilibria. On the other hand, we find that (D, R, A) is a 1-fault tolerant Nash equilibria, but not a 2-dominant strategy Nash equilibrium. The relationship between $(n-k)$-dominant strategy Nash equilibrium and k-fault tolerant Nash equilibrium is an open question when $1 \leq k \leq n-2$.

For each $k \in \{0,1,\ldots,n-1\}$ and each $u \in U$, let

$$kFTNE(\Gamma,u) \equiv \prod_{i\in I} \left\{ \bigcap_{K\subseteq I:\ |K|\leq k} kFTBR_i(\Gamma,u;s_{I\setminus\{K\cup\{i\}\}}) \right\}$$

be the set of k-fault tolerant Nash equilibria in the mechanism Γ at the profile of utility functions u. Note that $(n-1)FTNE(\Gamma,u) = DSE(\Gamma,u)$ and $0FTNE(\Gamma,u) = NE(\Gamma,u)$ for each $u \in U$.

3 Notion of Implementation

This section includes bounded rationality of the agents in implementation problems by introducing k-secure implementation in Subsection 3.3, which is an intermediate notion between dominant strategy implementation and secure implementation defined as double implementation in dominant strategy equilibria and Nash equilibria by Saijo, Sjöström, and Yamato (2007). In addition, in terms of the rationality of the agents, this section discusses the relationship between k-secure implementation and secure implementation and the relationship between k-secure implementation and k-fault tolerant implementation introduced by Eliaz (2002) in addition to the relationship with basic notion of implementation.

3.1 Basic Notion of Implementation

Similar to the notion of equilibrium, dominant strategy implementation and Nash implementation are basic notions of implementation. To show the relationship among such notions and k-secure implementation, this subsection introduces the two notions of implementation.

Dominant strategy implementation requires that for each profile of utility functions, (i) the outcome can be achieved by some dominant strategy equilibrium in the mechanism and (ii) each dominant strategy equilibrium in the mechanism achieves the outcome.

Definition 5. The mechanism Γ **implements the social choice function f in the dominant strategy equilibria** if and only if for each $u \in U$,

(i) there is $s \in DSE(\Gamma,u)$ such that $g(s) = f(u)$ and

(ii) for each $s' \in DSE(\Gamma,u)$, $g(s') = f(u)$.

The social choice function f **dominant strategy implementable** if and only if there is a mechanism Γ which implements f in the dominant strategy equilibria.

Nash implementation requires that for each profile of utility functions, (i) the outcome can be achieved by some Nash equilibrium in the mechanism and (ii) each Nash equilibrium in the mechanism achieves the outcome.

Definition 6. The mechanism Γ **implements the social choice function f in the Nash equilibria** if and only if for each $u \in U$,

(i) there is $s \in NE(\Gamma,u)$ such that $g(s) = f(u)$ and

(ii) for each $s' \in NE(\Gamma,u)$, $g(s') = f(u)$.

The social choice function f **Nash implementable** if and only if there is a mechanism Γ which implements f in the Nash equilibria.

By definition, dominant strategy implementation is independent of Nash implementation because dominant strategy implementation does not require Condition (ii) of Definition 6 and Nash implementation does not require Condition (i) of Definition 5.

3.2 Secure Implementation

Both dominant strategy implementation and Nash implementation assume that each agent is rational and chooses the agent's best response, especially the agent's dominant strategy when feasible. However, it is observed that there are boundedly rational agents who choose their best responses which are different from their dominant strategies in several experiments. Dominant strategy implementation does not require Condition (ii) of Definition 6. These imply the possibility that the mechanism which implements a socially optimal outcome in dominant strategy equilibria has a "bad" Nash equilibrium and a socially non-optimal outcome is achieved by the equilibrium when we actually use the mechanism. Saijo, Sjöström, and Yamato (2007) introduced **secure implementation** to solve this multiple equilibrium problem. Secure implementation is double implementation in dominant strategy equilibria and Nash equilibria, that is, for each profile of utility functions, (i) the outcome can be achieved by some dominant strategy equilibrium in the mechanism and (ii) each Nash equilibrium in the mechanism achieves the outcome.

Definition 7. The mechanism Γ **securely implements the social choice function** f if and only if for each $u \in U$,

(i) there is $s \in DSE(\Gamma, u)$ such that $g(s) = f(u)$ and

(ii) for each $s' \in NE(\Gamma, u)$, $g(s') = f(u)$.

The social choice function f **securely implementable** if and only if there is a mechanism Γ which securely implements f.

By definition, secure implementation is in general stronger than dominant strategy implementation and Nash implementation.

3.3 k-Secure Implementation

Secure implementation can be interpreted as a notion of implementation which requires that the mechanism successfully implements a socially optimal outcome even if all the agents are boundedly rational. On the other hand, in compensation for the robustness to bounded rationality of the agents, secure implementability is so strong and there is rarely a non-trivial securely implementable social choice function. To solve this difficulty, this paper introduces ***k*-secure implementation** which is not in general as strong as secure implementation in terms of the assumption about the number of boundedly rational agents. Given at most $k \in \{0, 1, \ldots, n\}$ boundedly rational agents, k-secure implementation requires that for each profile of utility functions, (i) the outcome can be achieved by some dominant strategy equilibrium in the mechanism and (ii) each $(n-k)$-dominant strategy Nash equilibrium in the mechanism achieves the outcome.

Definition 8. For each $k \in \{0,1,\ldots,n\}$, the mechanism Γ ***k*-securely implements the social choice function *f*** if and only if for each $u \in U$,

(i) there is $s \in DSE(\Gamma,u)$ such that $g(s) = f(u)$ and

(ii) for each $s' \in (n-k)DSNE(\Gamma,u)$, $g(s') = f(u)$.

For each $k \in \{0,1,\ldots,n\}$, the social choice function f ***k*-securely implementable** if and only if there is a mechanism Γ which k-securely implements f.

In Definition 8, there are at most $k \in \{0,1,\ldots,n\}$ boundedly rational agents in the mechanism. At each $(n-k)$-dominant strategy Nash equilibrium, although at least $(n-k)$ rational agents choose their dominant strategies, at most k boundedly rational agents might choose their best responses which are different from their dominant strategies (see Condition (ii) of Definition 8). Therefore, k-secure implementation is equivalent to dominant strategy implementation if $k = 0$ and secure implementation if $k = n$. If $1 \leq k \leq n-1$, then k-secure implementation is an intermediate notion between dominant strategy implementation and secure implementation. In addition, k-secure implementation is in general stronger than k'-secure implementation if $k < k'$ by definition.

3.4 Relationship between *k*-Secure Implementation and *k*-Fault Tolerant Implementation

As stated in Remark 1, each faulty agent does not necessarily choose the equilibrium strategy at the k-fault tolerant Nash equilibrium. This implies that a profile of strategies for all the agents is established as the result of the deviation from the k-fault tolerant Nash equilibrium by some faulty agent and the profile of strategies might be neither a dominant strategy equilibrium nor a Nash equilibrium. In consideration of such a situation, Eliaz (2002) introduced ***k*-fault tolerant implementation** which is robust to such a faultiness of the agents.

For each $s,s' \in S$, let $d(s,s') \equiv |\{i \in I \colon s_i \neq s'_i\}|$ be the number of agents whose strategies at s are different from those at s' respectively, called the **difference between s and s'**. In the framework of k-fault tolerant implementation, the difference conjectures the number of faulty agents. For each $k \in \{0,1,\ldots,n-1\}$ and each $s \in S$, let $D(k,s) \equiv \{s' \in S \colon d(s,s') \leq k\}$ be the set of profiles of strategies for the agents, whose difference from s is at most k, called the ***k*-neighborhood of *s***. In the framework of k-fault tolerant implementation, the neighborhood is the set of profiles of strategies for the agents, which are established by deviations from a k-fault tolerant Nash equilibrium, whose difference from the equilibrium is at most k. Given at most $k \in \{0,1,\ldots,n-1\}$ faulty agents, k-fault tolerant implementation requires that for each profile of utility functions, (i) the outcome can be achieved by some k-fault tolerant Nash equilibrium in the mechanism, (ii) each k-fault tolerant Nash equilibrium in the mechanism achieves the outcome, and (iii) each deviation from a k-fault tolerant Nash equilibrium in the mechanism, whose difference from the equilibrium is at most k, also achieves the outcome.

Definition 9. For each $k \in \{0,1,\ldots,n-1\}$, the mechanism Γ **implements the social choice function *f* in the *k*-fault tolerant Nash equilibria** if and only if for each $u \in U$,

(i) there is $s \in kFTNE(\Gamma,u)$ such that $g(s) = f(u)$,

(ii) for each $s' \in kFTNE(\Gamma, u)$, $g(s') = f(u)$, and

(iii) for each $s' \in kFTNE(\Gamma, u)$ and each $s'' \in D(k, s')$, $g(s'') = f(u)$.

For each $k \in \{0, 1, \ldots, n-1\}$, the social choice function f **k-fault tolerantly implementable** if and only if there is a mechanism Γ which implements f in the k-fault tolerant Nash equilibria.

In Definition 9, there are at most $k \in \{0, 1, \ldots, n-1\}$ faulty agents other than each rational agent in the mechanism. At each k-fault tolerant Nash equilibrium, although at least $(n-k)$ rational agents choose their best responses mutually, at most k faulty agents choose their any possible strategies. This implies that the profile of strategies $s'' \in D(k, s')$ might not be a Nash equilibrium. k-fault tolerant implementation requires that such a profile of strategies for the agents also achieves a socially optimal outcome by Condition (iii).

Remark 4. k-secure implementation assumes that each boundedly rational agent chooses the agent's equilibrium strategy. On the other hand, k-fault tolerant implementation assumes that each faulty agent does not necessarily choose the agent's equilibrium strategy. k-fault tolerant implementation is a robust notion of implementation in the sense that a socially optimal outcome is achieved even if faulty agents choose their any strategies.

By definition, k-fault tolerant implementation is equivalent to Nash implementation if $k = 0$ because $0FTNE(\Gamma, u) = NE(\Gamma, u)$ for each $u \in U$ and 0-fault tolerant implementation requires that $D(0, s') = \{s'\}$ for each $s' \in NE(\Gamma, u)$ and each $u \in U$. On the other hand, k-fault tolerant implementation is in general stronger than dominant strategy implementation if $k = n-1$ by definition because $(n-1)FTNE(\Gamma, u) = DSE(\Gamma, u)$ for each $u \in U$ and $(n-1)$-fault tolerant implementation requires that $g(s'') = f(u)$ for each $s'' \in D(n-1, s')$, each $s' \in DSE(\Gamma, u)$, and each $u \in U$, where s'' might not be the dominant strategy equilibrium. In addition, k-fault tolerant implementation is in general weaker than k'-fault tolerant implementation if $k < k'$ by definition.

Remark 5. By definition, 0-fault tolerant implementation is in general weaker than n-secure implementation because 0-fault tolerant implementation is equivalent to Nash implementation and n-secure implementation is equivalent to secure implementation. On the other hand, $(n-1)$-fault tolerant implementation is in general stronger than k-secure implementation for each $k \in \{0, 1, \ldots, n-1\}$ by definition because $(n-1)$-fault tolerant implementation requires dominant strategy implementation by Conditions (i) and (ii) of Definition 9 and that any deviation from the dominant strategy equilibrium by at most $(n-1)$ faulty agents also achieves a socially optimal outcome by Condition (iii) of Definition 9. [12]

4 Characterizations of Social Choice Functions

Although secure implementation is a solutions for the multiple equilibrium problem with dominant strategy implementation related to Nash equilibria, this notion is so strong and there is rarely a non-trivial securely implementable social choice function. In response to such a difficulty, the previous section introduced k-secure implementation which is an intermediate notion between dominant strategy implementation and secure implementation in terms of the rationality of the agents. This section introduces

[12] Note that such deviations include $(n-k)$-dominant strategy Nash equilibria.

the properties of social choice functions related to k-secure implementation and shows a characterization of k-securely implementable social choice functions by the associated direct revelation mechanism respectively. In addition, this section discusses the relationship among the properties related to dominant strategy implementation, secure implementation, and k-secure implementation.

4.1 Dominant Strategy Implementable Social Choice Functions

Dasgupta, Hammond, and Maskin (1979, Theorem 4.1.1) showed that **strategy-proofness** is a necessary condition for dominant strategy implementability by the associated direct revelation mechanism. Strategy-proofness requires that the truthful revelation is a dominant strategy in the direct revelation mechanism associated with a social choice function satisfying strategy-proofness.

Definition 10. The social choice function f satisfies **strategy-proofness** if and only if for each $u, u' \in U$ and each $i \in I$, $u_i(f(u_i, u'_{I\setminus\{i\}})) \geq u_i(f(u'_i, u'_{I\setminus\{i\}}))$.

Strategy-proofness does not require that truthful revelation is the "unique" dominant strategy. This implies the possibility that there is some agent's dominant strategy which is a the agent's untruthful revelation and induces a socially non-optimal outcome. To solve such a multiple equilibrium problem, Mizukami and Wakayama (2007, Corollary 1) and Saijo, Sjöström, and Yamato (2007, Theorem 3) independently characterized dominant strategy implementable social choice functions by strategy-proofness and the following condition, called **weak non-bossiness** (Mizukami and Wakayama, 2007; Saijo, Sjöström, and Yamato, 2007). [13] Weak non-bossiness requires that if the agent cannot change the agent's utility by the agent's revelation irrespective of other agents' revelations, then the outcome cannot change by the revelation in the direct revelation mechanism associated with a social choice function satisfying weak non-bossiness. [14]

Definition 11. The social choice function f satisfies **weak non-bossiness** if and only if for each $u, u' \in U$ and each $i \in I$, if

$$u_i(f(u_i, u''_{I\setminus\{i\}})) = u_i(f(u'_i, u''_{I\setminus\{i\}})) \text{ for each } u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}},$$

then $f(u_i, u'_{I\setminus\{i\}}) = f(u'_i, u'_{I\setminus\{i\}})$.

Under the strategy-proof mechanism, weak non-bossiness requires that if the agent has a dominant strategy which might be different from the agent's truthful revelation, then the outcome achieved by the dominant strategy equilibrium associated with such dominant strategies coincides with the outcome achieved by truthful revelations. This characteristic solves the multiple equilibrium problem with dominant strategy implementation related to dominant strategy equilibria.

Theorem 1 shows a characterization of dominant strategy implementable social choice functions by the associated direct revelation mechanism respectively.

Theorem 1 (Mizukami and Wakayama, 2007; Saijo, Sjöström, and Yamato, 2007)**.** *The social choice function is* ***dominant strategy implementable by the direct revelation mechanism associated with it*** *if and only if it satisfies* ***strategy-proofness*** *and* ***weak non-bossiness****.*

[13] Mizukami and Wakayama (2007) called weak non-bossiness by quasi-strong non-bossiness.

[14] Weak non-bossiness is a variant of non-bossiness (Satterthwaite and Sonnenschein, 1981). The social choice function f satisfies non-bossiness if and only if for each $u, u' \in U$ and each $i \in I$, if $f_i(u_i, u'_{I\setminus\{i\}}) = f_i(u'_i, u'_{I\setminus\{i\}})$, then $f(u_i, u'_{I\setminus\{i\}}) = f(u'_i, u'_{I\setminus\{i\}})$.

Remark 6. In general environments, Theorem 1 shows a characterization of dominant strategy implementable social choice functions by the associated "direct" revelation mechanism respectively, but not an "indirect" mechanism. In pure exchange economies, Mizukami and Wakayama (2007) showed an example of dominant strategy implementable social choice functions by an indirect mechanism, which satisfy strategy-proofness, but not weak non-bossiness.

4.2 Securely Implementable Social Choice Functions

By definition, strategy-proofness allows the existence of weakly dominant strategy in the associated direct revelation mechanism when the indifference in preferences is allowed. This implies the possibility that there is a Nash equilibrium which achieves a socially non-optimal outcome. Secure implementation requires that all Nash equilibria also achieve the socially optimal outcomes in the strategy-proof mechanism. Saijo, Sjöström, and Yamato (2007, Theorem 1) characterized securely implementable social choice functions by strategy-proofness and the following condition, called the **rectangular property** (Saijo, Sjöström, and Yamato, 2007). The rectangular property requires that if each agent cannot change the agent's utility by the agent's revelation, then the outcome cannot change by all the agents' revelations in the direct revelation mechanism associated with a social choice function satisfying the rectangular property.

Definition 12. The social choice function f satisfies the **rectangular property** if and only if for each $u, u' \in U$, if

$$u_i(f(u_i, u'_{I\setminus\{i\}})) = u_i(f(u'_i, u'_{I\setminus\{i\}})) \text{ for each } i \in I,$$

then $f(u) = f(u')$.

Saijo, Sjöström, and Yamato (2007, Proposition 2) showed that the rectangular property is in general stronger than weak non-bossiness. [15] Under the strategy-proof mechanism, the rectangular property requires that if each agent has the agent's best response which might be different from the agent's truthful revelation, then the outcome achieved by the Nash equilibrium associated with such best responses coincides with the outcome achieved by truthful revelations. This characteristic solves the multiple equilibrium problem in dominant strategy implementation related to Nash equilibria.

Theorem 2 shows a characterization of securely implementable social choice functions.

Theorem 2 (Saijo, Sjöström, and Yamato, 2007)**.** *The social choice function is* ***securely implementable*** *if and only if it satisfies* ***strategy-proofness*** *and the* ***rectangular property****.*

Saijo, Sjöström, and Yamato (2007, Corollary 1) showed another characterization of securely implementable social choice functions by strategy-proofness and the following conditions, called **strong non-bossiness** (Ritz, 1983) and the **outcome-rectangular property** (Saijo, Sjöström, and Yamato, 2007). [16]

[15]Precisely, Saijo, Sjöström, and Yamato (2007, Proposition 2) showed that the rectangular property is in general stronger than a stronger variant of weak non-bossiness, called strong non-bossiness, defined in Definition 13 presented here.

[16]Ritz (1983) called strong non-bossiness by non-corruptibility and Saijo, Sjöström, and Yamato (2007) by non-bossiness. See Ritz (1983), Nicolò (2004), Mizukami and Wakayama (2005), Saijo, Sjöström, and Yamato (2007), Berga and Moreno (2009), and Nishizaki (2012, 2014, 2018a) for theoretical results on strong non-bossiness under strategy-proof mechanisms. Mizukami and Wakayama (2017, Theorem 1) also showed an alternative characterization of securely implementable social choice functions by individual maximality (Mizukami and Wakayama, 2017) and restricted monotonicity (Mizukami and

Strong non-bossiness requires that if the agent cannot change the agent's utility by the agent's revelation, then the outcome cannot change by the revelation in the direct revelation mechanism associated with a social choice function satisfying strong non-bossiness. The outcome-rectangular property requires that if each agent cannot change the outcome by the agent's revelation, then the outcome cannot change by all the revelations in the direct revelation mechanism associated with a social choice function satisfying the outcome-rectangular property.

Definition 13. The social choice function f satisfies **strong non-bossiness** if and only if for each $u, u' \in U$ and each $i \in I$, if

$$u_i(f(u_i, u'_{I\setminus\{i\}})) = u_i(f(u'_i, u'_{I\setminus\{i\}})),$$

then $f(u_i, u'_{I\setminus\{i\}}) = f(u'_i, u'_{I\setminus\{i\}})$.

Definition 14. The social choice function f satisfies the **outcome-rectangular property** if and only if for each $u, u' \in U$, if $f(u_i, u'_{I\setminus\{i\}}) = f(u'_i, u'_{I\setminus\{i\}})$ for each $i \in I$, then $f(u) = f(u')$.

By definition, strong non-bossiness is in general stronger than weak non-bossiness and non-bossiness. In addition, the outcome-rectangular property is in general weaker than the rectangular property by definition.

Theorem 3 (Saijo, Sjöström, and Yamato, 2007)**.** *The social choice function is **securely implementable** if and only if it satisfies **strategy-proofness**, **strong non-bossiness**, and the **outcome-rectangular property**.*

Saijo, Sjöström, and Yamato (2007) also showed the revelation principle for secure implementation as Theorem 4, which allows us to restrict our attention to the direct revelation mechanism associated with a securely implementable social choice function when we consider secure implementation.

Theorem 4 (Saijo, Sjöström, and Yamato, 2007)**.** *The social choice function is **securely implementable** if and only if the direct revelation mechanism associated with it **securely implements it**.*

4.3 k-Securely Implementable Social Choice Functions

Although secure implementation solves the multiple equilibrium problem with dominant strategy implementation related to Nash equilibria, it is difficult to securely implement a socially optimal outcome. This paper attempts to solve this difficulty in terms of the assumption about the number of boundedly rational agents. As stated above, the rectangular property requires that if each agent has the agent's best response which might be different from the agent's truthful revelation, then the outcome achieved by the Nash equilibrium associated with such best responses coincides with the outcome achieved by truthful revelations in the strategy-proof mechanism. In other words, secure implementation requires that the socially optimal outcome is achieved even if all the agents are boundedly rational. k-secure implementation is not as strong as secure implementation in terms of the assumption about the number of boundedly rational agents, that is, k-secure implementation requires that the socially optimal outcome is achieved if at most $k \in \{0, 1, \ldots, n\}$ agents are boundedly rational. This paper characterizes k-securely implementable social choice functions by strategy-proofness and the following condition, called the ***k*-rectangular property**,

Wakayama, 2017). In general, individual maximality is weaker than strategy-proofness and restricted monotonicity is stronger than Maskin monotonicity (Maskin, 1977, 1999) by definition, which is a necessary condition for Nash implementability.

defined as a "convex combination" of weak non-bossiness which is a necessary condition for dominant strategy implementability by the associated direct revelation mechanism and the rectangular property which is a necessary condition for secure implementability.

Definition 15. For each $k \in \{0, 1, \ldots, n\}$, the social choice function f satisfies the ***k*-rectangular property** if and only if for each $u, u' \in U$, each $K \subseteq I$ with $|K| \leq k$, and each $i \in I \setminus K$, if

(i) $u_i(f(u_i, u''_{I\setminus\{i\}})) = u_i(f(u'_i, u''_{I\setminus\{i\}}))$ for each $u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}$ and

(ii) $u_j(f(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) = u_j(f(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}))$ for each $j \in K$,

then $f(u_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$. [17]

In Definition 15, we can consider that K is the set of boundedly rational agents and $I \setminus K$ is the set of rational agents. In the direct revelation mechanism associated with a social choice function satisfying strategy-proofness and the k-rectangular property, truthful revelation is a dominant strategy for each agent. Condition (i) supposes that u'_i is also a dominant strategy for rational agent $i \in I \setminus K$. In addition, Condition (ii) supposes that u'_j is a best response for each boundedly rational agent $j \in K$ to the profile of strategies for other agents $(u'_{K\setminus\{j\}}, u'_{I\setminus K})$. The k-rectangular property requires that the outcome does not change by such dominant strategy and best responses.

If $k = 0$, then $K = \emptyset$ and $I \setminus K = I$ in Definition 15. This implies that Condition (ii) of Definition 15 is redundant and the 0-rectangular property is equivalent to weak non-bossiness by definition. Together with Theorem 1, this implies the following characterization of dominant strategy implementable social choice functions by the associated direct revelation mechanism respectively.

Corollary 1. *The social choice function is* ***dominant strategy implementable by the direct revelation mechanism associated with it*** *if and only if it satisfies* ***strategy-proofness*** *and the* ***0-rectangular property****.*

If $k = 1$, then $K \neq \emptyset$ and $I \setminus K \subset I$ when $|K| = 1$. On the basis of this relationship, Proposition 1 shows that the 1-rectangular property is equivalent to strong non-bossiness. By definition, this implies that the 1-rectangular property is in general stronger than weak non-bossiness which is equivalent to the 0-rectangular property.

Proposition 1. *The social choice function f satisfies the* ***1-rectangular property*** *if and only if f satisfies* ***strong non-bossiness****.*

Proof. See Appendix. □

By Theorem 3 and Proposition 1, we have the following characterization of securely implementable social choice functions.

Corollary 2. *The social choice function is* ***securely implementable*** *if and only if it satisfies* ***strategy-proofness****, the* ***1-rectangular property****, and the* ***outcome-rectangular property****.*

[17] Note that $K \setminus \{i\} = K$ when $|K| < n$ because $i \in I \setminus K$. In addition, $K \cup \{i\} = I$ when $|K| = n$ because $K = I$.

If $k = n$, then $K \neq \emptyset$ and $I \setminus K \subset I$ when $|K| \geq 1$, especially $I \setminus K = \emptyset$ when $|K| = n$. On the basis of this relationship, Proposition 2 shows that the n-rectangular property is equivalent to the rectangular property by an argument similar to Proposition 1. Together with Proposition 2 of Saijo, Sjöström, and Yamato (2007), this implies that the n-rectangular property is in general stronger than strong non-bossiness which is equivalent to the 1-rectangular property by Proposition 1.

Proposition 2. *The social choice function f satisfies the **n-rectangular property** if and only if f satisfies the **rectangular property**.*

Proof. See Appendix. □

By Theorem 2 and Proposition 2, we have the following characterization of securely implementable social choice functions.

Corollary 3. *The social choice function is **securely implementable** if and only if it satisfies **strategy-proofness** and the **n-rectangular property**.*

On the basis of Proposition 1, we know that the 1-rectangular property is in general stronger than the 0-rectangular property. In addition, on the basis of Proposition 2 and Proposition 2 of Saijo, Sjöström, and Yamato (2007), we know that the n-rectangular property is in general stronger than the 1-rectangular property. Essentially, the k-rectangular property is stronger with the increase in the number of k. [18]

Under the strategy-proof mechanism, the k-rectangular property requires that if (i) the rational agent has the agent's dominant strategy which might be different from the agent's truthful revelation and (ii) each boundedly rational agent, where there are at most $k \in \{0, 1, \ldots, n\}$ boundedly rational agents, has the agent's best response which might be different from the agent's truthful revelation, then the outcome achieved by the $(n-k)$-dominant strategy Nash equilibrium associated with such dominant strategies and best responses coincides with the outcome achieved by truthful revelations. This characteristic solves the difficulty of secure implementation in terms of the assumption about the number of boundedly rational agents.

Because k-secure implementation implies dominant strategy implementation by definition, Corollary 4 shows that strategy-proofness is a necessary condition for k-secure implementability by the associated direct revelation mechanism on the basis of Theorem 1.

Corollary 4. *For each $k \in \{0, 1, \ldots, n\}$, if the social choice function is **k-securely implementable by the direct revelation mechanism associated with it**, then it satisfies **strategy-proofness**.*

In addition, Proposition 3 shows that the k-rectangular property is a necessary condition for k-secure implementability by the associated direct revelation mechanism.

Proposition 3. *For each $k \in \{0, 1, \ldots, n\}$, if the social choice function f is **k-securely implementable by the direct revelation mechanism associated with it Γ^f**, then f satisfies the **k-rectangular property**.*

Proof. Let $u, u' \in U$, $K \subseteq I$ with $|K| \leq k$, and $i \in I \setminus K$ be such that (i) $u_i(f(u_i, u''_{I\setminus\{i\}})) = u_i(f(u'_i, u''_{I\setminus\{i\}}))$ for each $u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}$ and (ii) $u_j(f(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) = u_j(f(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}))$ for each $j \in K$. This proof consists of the following four claims.

[18] For each $k, k' \in \{0, 1, \ldots, n\}$, if $k < k'$, then the k'-rectangular property is in general stronger than the k-rectangular property by definition because we can take $K' \subseteq I$ with $|K'| \leq k'$ such that $K' = K$, where $K \subseteq I$ with $|K| \leq k$.

Claim 1. $u_i' \in DS_i(\Gamma, u_i)$.

By Corollary 4, we know that $u_i \in DS_i(\Gamma^f, u_i)$. This implies that

$$u_i(g^f(u_i, u''_{I\setminus\{i\}})) \geq u_i(g^f(u''_i, u''_{I\setminus\{i\}})) \text{ for each } u''_i \in U_i \text{ and each } u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}. \tag{1}$$

By Corollary 4, we also know that $u'_i \in DS_i(\Gamma^f, u'_i)$ and $u''_{I\setminus\{i\}} \in DS_{I\setminus\{i\}}(\Gamma^f, u''_{I\setminus\{i\}})$ for each $u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}$. This implies that $(u_i, u''_{I\setminus\{i\}}) \in DS(\Gamma^f, (u_i, u''_{I\setminus\{i\}}))$ and $(u'_i, u''_{I\setminus\{i\}}) \in DS(\Gamma^f, (u'_i, u''_{I\setminus\{i\}}))$ for each $u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}$. Because f is k-securely implementable by Γ^f, these imply that $g^f(u_i, u''_{I\setminus\{i\}}) = f(u_i, u''_{I\setminus\{i\}})$ and $g^f(u'_i, u''_{I\setminus\{i\}}) = f(u'_i, u''_{I\setminus\{i\}})$ for each $u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}$. Together with Condition (i), this implies that

$$u_i(g^f(u_i, u''_{I\setminus\{i\}})) = u_i(g^f(u'_i, u''_{I\setminus\{i\}})) \text{ for each } u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}. \tag{2}$$

By (1) and (2), we find that $u_i(g^f(u'_i, u''_{I\setminus\{i\}})) \geq u_i(g^f(u''_i, u''_{I\setminus\{i\}}))$ for each $u''_i \in U_i$ and each $u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}$, that is, $u'_i \in DS_i(\Gamma^f, u_i)$.

Claim 2. $u'_K \in BR_K(\Gamma^f, u_K | u'_{I\setminus K})$.

By Corollary 4, we know that $u_j \in DS_j(\Gamma^f, u_j)$ for each $j \in K$. This implies that for each $j \in K$,

$$u_j(g^f(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) \geq u_j(g^f(u''_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) \text{ for each } u''_j \in U_j. \tag{3}$$

By Corollary 4, we also know that $u'_j \in DS_j(\Gamma^f, u'_j)$ and $u'_{K\setminus\{j\}} \in DS_{K\setminus\{j\}}(\Gamma^f, u'_{K\setminus\{j\}})$ for each $j \in K$ and $u'_{I\setminus K} \in DS_{I\setminus K}(\Gamma^f, u'_{I\setminus K})$. This implies that $(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}) \in DS(\Gamma^f, (u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}))$ and $(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}) \in DS(\Gamma^f, (u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}))$ for each $j \in K$. Because f is k-securely implementable by Γ^f, these imply $g^f(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}) = f(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})$ and $g^f(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}) = f(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})$ for each $j \in K$. Together with Condition (ii), this implies that for each $j \in K$,

$$u_j(g^f(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) = u_j(g^f(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})). \tag{4}$$

By (3) and (4), we find that for each $j \in K$, $u_j(g^f(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) \geq u_j(g^f(u''_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}))$ for each $u''_j \in U_j$. This implies that $u'_j \in BR_j(\Gamma^f, u_j | u'_{K\setminus\{j\}}, u'_{I\setminus K})$ for each $j \in K$, that is, $u'_K \in BR_K(\Gamma^f, u_K | u'_{I\setminus K})$.

Claim 3. $f(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$.

By Corollary 4, we know that $u'_i \in DS_i(\Gamma^f, u'_i)$ and $u'_{I\setminus\{K\cup\{i\}\}} \in DS_{I\setminus\{K\cup\{i\}\}}(\Gamma^f, u'_{I\setminus\{K\cup\{i\}\}})$. By Claim 2, we know that $u'_K \in BR_K(\Gamma^f, u_K | u'_{I\setminus K})$. These imply that

$$(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) \in (n-k)DSNE(\Gamma^f, (u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})).$$

Because f is k-securely implementable by Γ^f, this implies that

$$g^f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}).$$

Because $(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) \in DSE(\Gamma^f, (u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}))$ by Corollary 4 and

$$g^f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$$

by k-secure implementability, this implies that $f(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$.

Claim 4. $f(u_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$.

By Corollary 4, we know that $u_K \in DS_K(\Gamma^f, u_K)$ and $u'_{I\setminus\{K\cup\{i\}\}} \in DS_{I\setminus\{K\cup\{i\}\}}(\Gamma^f, u'_{I\setminus\{K\cup\{i\}\}})$. By Claim 1, we know that $u'_i \in DS_i(\Gamma^f, u_i)$. These imply that $(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) \in DS(\Gamma^f, (u_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}))$. Because f is k-securely implementable by Γ^f, this implies that

$$g^f(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}).$$

Because $(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) \in DSE(\Gamma^f, (u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}))$ by Corollary 4 and

$$g^f(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$$

by k-secure implementability, this implies that $f(u_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$. Together with Claim 3, this implies that $f(u_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$. □

On the other hand, Proposition 4 shows that the combination of strategy-proofness and the k-rectangular property is a sufficient condition for k-secure implementability by the associated direct revelation mechanism.

Proposition 4. *For each $k \in \{0, 1, \ldots, n\}$, if the social choice function f satisfies* ***strategy-proofness*** *and the* ***k-rectangular property****, then the direct revelation mechanism associated with it Γ^f* ***k-securely implements f****.*

Proof. Let $u \in U$. By the definition of Γ^f and **strategy-proofness**, we find that $u \in DSE(\Gamma^f, u)$ and $g^f(u) = f(u)$. The remainder of this proof confirms that $g^f(u') = f(u)$ for each $u' \in (n-k)DSNE(\Gamma^f, u)$ according to the number of k. If $k = 0$, then we know that $(n-k)DSNE(\Gamma^f, u) = DSE(\Gamma^f, u)$ and the k-rectangular property is equivalent to weak non-bossiness by definition. Together with Theorem 1, this implies that $g^f(u') = f(u)$ for each $u' \in (n-k)DSNE(\Gamma^f, u)$. If $k = n$, then we know that $(n-k)DSNE(\Gamma^f, u) = NE(\Gamma^f, u)$ and the k-rectangular property is equivalent to the rectangular property by definition. Together with Theorem 2, this implies that $g^f(u') = f(u)$ for each $u' \in (n-k)DSNE(\Gamma^f, u)$. In what follows, we consider the case in which $0 < k < n$.

Let $u' \in (n-k)DSNE(\Gamma^f, u)$. If $0 < k < n$, then we know that $I \setminus K \neq \emptyset$ and $K \neq \emptyset$. Let $i_1 \in I \setminus K$. By definition, we know that $u'_{i_1} \in DS_{i_1}(\Gamma^f, u)$. Together with the definition of Γ^f and **strategy-proofness**, this implies that

$$u_{i_1}(f(u_{i_1}, u''_{I\setminus\{i_1\}})) = u_{i_1}(f(u'_{i_1}, u''_{I\setminus\{i_1\}})) \text{ for each } u''_{I\setminus\{i_1\}} \in U_{I\setminus\{i_1\}}. \tag{5}$$

By definition, we also know that $u'_j \in BR_j(\Gamma^f, u; u'_{K\setminus\{j\}}, u'_{I\setminus K})$ for each $j \in K$. Together with the definition of Γ^f and **strategy-proofness**, this implies that

$$u_j(f(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) = u_j(f(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) \text{ for each } j \in K. \tag{6}$$

By (5), (6), and the ***k*-rectangular property**, we find that

$$f(u_{i_1}, u_{K\setminus\{i_1\}}, u'_{I\setminus\{K\cup\{i_1\}\}}) = f(u'_{i_1}, u'_{K\setminus\{i_1\}}, u'_{I\setminus\{K\cup\{i_1\}\}}). \tag{7}$$

Let $u^* \equiv (u_{i_1}, u_{K\setminus\{i_1\}}, u'_{I\setminus\{K\cup\{i_1\}\}})$ and $i_2 \in I \setminus \{K \cup \{i_1\}\}$. By definition, we know that $u^*_{i_2} \in DS_{i_2}(\Gamma^f, u)$. Together with the definition of Γ^f and **strategy-proofness**, this implies that

$$u_{i_2}(f(u_{i_2}, u''_{I\setminus\{i_2\}})) = u_{i_2}(f(u^*_{i_2}, u''_{I\setminus\{i_2\}})) \text{ for each } u''_{I\setminus\{i_2\}} \in U_{I\setminus\{i_2\}}. \tag{8}$$

Together with the definition of Γ^f and the definition of u^*, we know that

$$u_j(f(u_j, u^*_{K\setminus\{j\}}, u^*_{I\setminus K})) = u_j(f(u^*_j, u^*_{K\setminus\{j\}}, u^*_{I\setminus K})) \text{ for each } j \in K. \tag{9}$$

By (8), (9), and the ***k*-rectangular property**, we find that $f(u_{i_2}, u_{K\setminus\{i_2\}}, u^*_{I\setminus\{K\cup\{i_2\}\}}) = f(u^*_{i_2}, u^*_K, u^*_{I\setminus\{K\cup\{i_2\}\}})$, that is,

$$f(u_{i_1}, u_{i_2}, u_{K\setminus\{i_2\}}, u'_{I\setminus\{K\cup\{i_1,i_2\}\}}) = f(u_{i_1}, u'_{i_2}, u_{K\setminus\{i_2\}}, u'_{I\setminus\{K\cup\{i_1,i_2\}\}}). \tag{10}$$

By (7) and (10), we find that $f(u_{i_1}, u_{i_2}, u_{K\setminus\{i_2\}}, u'_{I\setminus\{K\cup\{i_1,i_2\}\}}) = f(u'_{i_1}, u'_{i_2}, u'_{K\setminus\{i_2\}}, u'_{I\setminus\{K\cup\{i_1,i_2\}\}})$. By sequentially replacing u'_i by u_i for each $i \in I\setminus\{K\cup\{i_1,i_2\}\}$ in this manner, we find that $f(u) = f(u')$, that is, $g^f(u') = f(u)$ by the definition of Γ^f. □

By Corollary 4 and Propositions 3 and 4, we have the following characterization of *k*-securely implementable social choice functions by the associated direct revelation mechanism respectively.

Theorem 5. *For each $k \in \{0,1,\dots,n\}$, the social choice function is* ***k-securely implementable by the direct revelation mechanism associated with it*** *if and only if it satisfies* ***strategy-proofness*** *and the* ***k-rectangular property****.*

Remark 7. In general environments, Theorem 5 shows a characterization of *k*-securely implementable implementable social choice functions by the associated "direct" revelation mechanism respectively, but not an "indirect" mechanism. If $k = 0$, then we know that there is a *k*-securely implementable implementable social choice function by an indirect mechanism, which satisfies strategy-proofness, but not the *k*-rectangular property, on the basis of the example of Mizukami and Wakayama (2007) in pure exchange economies. If $k = n$, then we know that Theorem 5 also shows a characterization of *k*-securely implementable implementable social choice functions by an indirect mechanism on the basis of Theorems 2 and 4. It is an open question whether the revelation principle for *k*-secure implementation similar to Theorem 4 is established when $1 \le k \le n-1$.

5 Application: Binary Voting

To show a possibility of *k*-secure implementation, we consider the following binary voting environment. Let n be an odd integer and $A = \{0,1\}$. For each $i \in I$, let $U_i = \{u^0_i, u^1_i\}$, where $u^0_i(0) > u^0_i(1)$ and $u^1_i(1) > u^1_i(0)$. In this environment, it is well-known that majority rule is non-dictatorial, Pareto-efficient, and dominant strategy implementable. For each $u \in U$, let $n^0(u) \in \{0,\dots,n\}$ be the number of agents who prefer 0 to 1 and $n^1(u) \equiv n - n^0(u)$ be the number of agents who prefer 1 to 0 at u. The direct revelation mechanism Γ^f is **majority rule** if and only if for each $u \in U$,

$$g^f(u) = \begin{cases} 0 & \text{if } n^0(u) > n^1(u), \\ 1 & \text{if } n^0(u) < n^1(u). \end{cases}$$

Saijo, Sjöström, and Yamato (2007) showed that majority rule is not securely implementable. In fact, Example 3 shows this impossibility.

Example 3. We consider the five-person binary voting environment. Let Γ^f be majority rule. Because the associated social choice function f is dominant strategy implementable by Γ^f, we know that $(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0) \in DS(\Gamma^f,(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0))$ and $g^f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0) = f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0)$. On the other hand, we know that $(u_1^1,u_2^1,u_3^1,u_4^1,u_5^1) \in NE(\Gamma^f,(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0))$. Secure implementation requires that $g^f(u_1^1,u_2^1,u_3^1,u_4^1,u_5^1) = f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0)$ and we find that $g^f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0) = g^f(u_1^1,u_2^1,u_3^1,u_4^1,u_5^1)$. This is a contradiction.

By definition, we know that majority rule is 0-securely implementable. In addition, majority rule is 1-securely implementable because any best response for a boundedly rational agent does not change the outcome. Example 3 implies that majority rule is not 5-securely implementable when $n=5$. In what follows, we consider 2-securely implementability of majority rule at $(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0)$ and $(u_1^0,u_2^0,u_3^0,u_4^0,u_5^1)$.

Case 1. $(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0)$.

In this case, we know that $(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0) \in DS(\Gamma^f,(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0))$ and $g^f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0) = f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0)$. We suppose that agents 3 and 4 are boundedly rational and might choose their best responses u_3^1 and u_4^1 at $(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0)$. If both the agents choose their best responses respectively, then we find that $(u_1^0,u_2^0,u_3^1,u_4^1,u_5^0) \in 3DSNE(\Gamma^f,(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0))$. 2-secure implementation requires that $g^f(u_1^0,u_2^0,u_3^1,u_4^1,u_5^0) = f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0)$ and we find that $g^f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^0) = g^f(u_1^0,u_2^0,u_3^1,u_4^1,u_5^0)$. This is "not" a contradiction.

Case 2. $(u_1^0,u_2^0,u_3^0,u_4^0,u_5^1)$.

In this case, we know that $(u_1^0,u_2^0,u_3^0,u_4^0,u_5^1) \in DS(\Gamma^f,(u_1^0,u_2^0,u_3^0,u_4^0,u_5^1))$ and $g^f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^1) = f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^1)$. Similar to Case 1, we suppose that agents 3 and 4 are boundedly rational and might choose their best responses u_3^1 and u_4^1 at $(u_1^0,u_2^0,u_3^0,u_4^0,u_5^1)$. If both the agents choose their best responses respectively, then we find that $(u_1^0,u_2^0,u_3^1,u_4^1,u_5^1) \in 3DSNE(\Gamma^f,(u_1^0,u_2^0,u_3^0,u_4^0,u_5^1))$. 2-secure implementation requires that $g^f(u_1^0,u_2^0,u_3^1,u_4^1,u_5^1) = f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^1)$ and we find that $g^f(u_1^0,u_2^0,u_3^0,u_4^0,u_5^1) = g^f(u_1^0,u_2^0,u_3^1,u_4^1,u_5^1)$. This is a contradiction.

The major difference between Cases 1 and 2 depends on whether they satisfy the condition $n^0(u) > (n/2)+k$. In Case 1, we find that $n^0((u_1^0,u_2^0,u_3^0,u_4^0,u_5^0)) = 5 > 4.5 = (5/2)+2$ and the condition is satisfied. On the other hand, in Case 2, we find that $n^0((u_1^0,u_2^0,u_3^0,u_4^0,u_5^1)) = 4 < 4.5 = (5/2)+2$ and the condition is not satisfied. On the basis of the above argument, Theorem 6 shows a condition for k-secure implementability of majority rule.

Theorem 6. *For each $k \in \{0,1,\dots,\lfloor n/2 \rfloor\}$, majority rule is k-securely implementable when the domain of the rule is $\{u \in U\colon n^0(u) > (n/2)+k$ or $n^1(u) > (n/2)+k\}$.* [19]

Proof. Because it is well-known that majority rule satisfies strategy-proofness, we confirm that it also satisfies the k-rectangular property according to Theorem 5. Let f be the associated social choice function. In addition, let $u,u' \in \{u \in U\colon n^0(u) > (n/2)+k$ or $n^1(u) > (n/2)+k\}$, $K \subseteq I$ with $|K| \leq k$, and $i \in I \setminus K$ be such that (i) $u_i(f(u_i,u''_{I\setminus\{i\}})) = u_i(f(u'_i,u''_{I\setminus\{i\}}))$ for each $u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}$ and (ii) $u_j(f(u_j,u'_{K\setminus\{j\}},u'_{I\setminus K})) = u_j(f(u'_j,u'_{K\setminus\{j\}},u'_{I\setminus K}))$ for each $j \in K$. By Condition (i) and the definition of the set of utility functions for each agent, we find that

$$u_i = u'_i. \tag{11}$$

[19] $\lfloor \cdot \rfloor$ is the floor function.

We prove this theorem according to the number of k.

If $k=0$, then $K=\emptyset$ and Condition (ii) is redundant. By (11), we know that $f(u_i,u'_{I\setminus\{i\}})=f(u'_i,u'_{I\setminus\{i\}})$. This implies that f satisfies the k-rectangular property.

If $1\leq k\leq \lfloor n/2\rfloor$, then $K\neq\emptyset$ and Condition (ii) is not redundant. [20] We consider the case in which $n^0(u')>(n/2)+k$. In this case, we know that

$$f(u'_i,u'_{K\setminus\{i\}},u'_{I\setminus\{K\cup\{i\}\}})=0. \tag{12}$$

By the definition of the domain of f, we find that

$$f(u'_i,u_{K\setminus\{i\}},u'_{I\setminus\{K\cup\{i\}\}})=0 \tag{13}$$

even if $k=\lfloor n/2\rfloor$, $|K|=k$, and $u_j=u^1_j$ for each $j\in K$ because $n^0(u'_j,u_{K\setminus\{j\}},u'_{I\setminus K})\geq n^0(u')-k>(n/2)+k-k=(n/2)$. By (12) and (13), we find that $f(u'_i,u_{K\setminus\{i\}},u'_{I\setminus\{K\cup\{i\}\}})=f(u'_i,u'_{K\setminus\{i\}},u'_{I\setminus\{K\cup\{i\}\}})$. Together with (11), this implies that $f(u_i,u_{K\setminus\{i\}},u'_{I\setminus\{K\cup\{i\}\}})=f(u'_i,u'_{K\setminus\{i\}},u'_{I\setminus\{K\cup\{i\}\}})$. □

In fact, both sets of outcomes achieved by dominant strategy equilibria and $(n-k)$-dominant strategy Nash equilibria in majority rule respectively coincide with the set of outcomes assigned by the associated social choice function when the number of boundedly rational agents is less than half the number of all the agents and the domain of the rule satisfies the condition supposed in Theorem 6. Let $U^0\equiv\{u\in U\colon n^0(u)>(n/2)+k\}$ and $u\in U^0$. Because majority rule satisfies strategy-proofness, we know that $u\in DSE(\Gamma^f,u)$ and $g^f(u)=f(u)$. Let $u'\in(n-k)DSNE(\Gamma^f,u)$. Because $u\in U^0$, we find that $n^0(u')\geq n^0(u)-k>n/2>n^1(u')$. This implies that $g^f(u')=0$ and we find that $g^f(u')=f(u)$ because $u\in U^0$ and $f(u)=0$. By an argument similar to the case of U^0, we have the same result when we take $u\in U^1\equiv\{u\in U\colon n^1(u)>(n/2)+k\}$. This implies that majority rule k-securely implements a socially optimal outcome under the assumption of Theorem 6. This possibility result suggests that majority rule works well in practice when the number of boundedly rational agents is sufficiently small.

6 Conclusion

Under the mechanism in which there is a dominant strategy equilibrium at the profile of utility functions, implementation theory mostly assumes that all the agents are rational and they choose their dominant strategies. Although this assumption is reasonable when the indifference in preferences is not allowed for each agent, it is questionable whether this assumption is reasonable when the indifference in preferences is allowed for some agent because there might be the agent's best response which is different from the agent's dominant strategy. In addition, it is observed that several subjects do not choose their dominant strategies even though they can choose them in the experiment. Alternatively, some of them choose their best responses which are different from their dominant strategies. On the basis of such theoretical question and experimental results, this paper assumes that there might be several boundedly rational agents who choose their best responses which are different from their dominant strategies. Under the assumption, this paper introduced k-secure implementation and showed necessary and sufficient conditions for k-secure implementability by the associated direct revelation mechanism.

[20] That being said, we need to pay attention to Condition (ii) because this condition is satisfied on its own on the domain supposed in this theorem. By the definition of the domain of f, we find that $n^a(u_j,u'_{K\setminus\{j\}},u'_{I\setminus K})\geq n^a(u')-1>(n/2)+k-1\geq(n/2)$ for $a=0,1$. This implies that $f(u_j,u'_{K\setminus\{j\}},u'_{I\setminus K})=f(u'_j,u'_{K\setminus\{j\}},u'_{I\setminus K})$ for each $j\in K$.

k-secure implementation is in general weaker than secure implementation if $k \leq n-1$ by definition. This implies the possibility that there is a non-trivial social choice function which is not securely implementable, but k-securely implementable by the associated direct revelation mechanism. In fact, Theorem 6 showed that majority rule is not securely implementable, but k-securely implementable when the number of boundedly rational agents is sufficiently small. Therefore, it is interesting to investigate non-trivial k-securely implementable social choice functions in various environments. In addition, experimental studies on k-secure implementation are important. These research issues are left for future works.

Appendix: Relationship among the k-Rectangular Property, Strong Non-Bossiness, and the Rectangular Property

As stated above, Proposition 1 in Subsection 4.3 shows that the 1-rectangular property is equivalent to strong non-bossiness. By definition, this implies that the 1-rectangular property is in general stronger than weak non-bossiness which is equivalent to the 0-rectangular property.

Proposition 1. *The social choice function f satisfies the* ***1-rectangular property*** *if and only if f satisfies* ***strong non-bossiness****.*

Proof. This proof consists of the following two claims.

Claim 1. *f satisfies the* ***1-rectangular property*** *if f satisfies* ***strong non-bossiness****.*

Let $u, u' \in U$, $K \subseteq I$ with $|K| \leq 1$, and $i \in I \setminus K$ be such that (i) $u_i(f(u_i, u''_{I\setminus\{i\}})) = u_i(f(u'_i, u''_{I\setminus\{i\}}))$ for each $u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}$ and (ii) $u_j(f(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) = u_j(f(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}))$ for each $j \in K$. We prove this claim according to the number of $|K|$.

If $|K| = 0$, then $K = \emptyset$ and $I \setminus K = I$. This implies that Condition (ii) is redundant and it is sufficient to show that $f(u_i, u'_{I\setminus\{i\}}) = f(u'_i, u'_{I\setminus\{i\}})$ for $i \in I$ on the basis of Condition (i). Let $u''_{I\setminus\{i\}} = u'_{I\setminus\{i\}}$. Together with Condition (i), this implies that $u_i(f(u_i, u'_{I\setminus\{i\}})) = u_i(f(u'_i, u'_{I\setminus\{i\}}))$. Together with **strong non-bossiness**, this implies that $f(u_i, u'_{I\setminus\{i\}}) = f(u'_i, u'_{I\setminus\{i\}})$.

If $|K| = 1$, then $K \neq \emptyset$ and there is $j \in K$. This implies that it is sufficient to show that $f(u_i, u_j, u'_{I\setminus\{i,j\}}) = f(u'_i, u'_j, u'_{I\setminus\{i,j\}})$ on the basis of Conditions (i) and (ii). Because $K = \{j\}$, Condition (ii) can be rewritten as follows: $u_j(f(u_j, u'_{I\setminus\{j\}})) = u_j(f(u'_j, u'_{I\setminus\{j\}}))$. Together with **strong non-bossiness**, this implies that

$$f(u'_i, u_j, u'_{I\setminus\{i,j\}}) = f(u'_i, u'_j, u'_{I\setminus\{i,j\}}). \tag{14}$$

Let $u''_{I\setminus\{i\}} = (u_j, u'_{I\setminus\{i,j\}})$. Together with Condition (i), this implies that $u_i(f(u_i, u_j, u'_{I\setminus\{i,j\}})) = u_i(f(u'_i, u_j, u'_{I\setminus\{i,j\}}))$. Together with **strong non-bossiness**, this implies that

$$f(u_i, u_j, u'_{I\setminus\{i,j\}}) = f(u'_i, u_j, u'_{I\setminus\{i,j\}}). \tag{15}$$

By (14) and (15), we find that $f(u_i, u_j, u'_{I\setminus\{i,j\}}) = f(u'_i, u'_j, u'_{I\setminus\{i,j\}})$.

Claim 2. *f satisfies the* ***1-rectangular property*** *only if f satisfies* ***strong non-bossiness****.*

Let $u, u' \in U$ and $j \in I$ be such that

$$u_j(f(u_j, u'_{I\setminus\{j\}})) = u_j(f(u'_j, u'_{I\setminus\{j\}})). \tag{16}$$

In addition, let $K \equiv \{j\}$, $i \in I \setminus K$, and $u^*_i \equiv u'_i$. Because $u^*_i \equiv u'_i$, we find that

$$u^*_i(f(u^*_i, u''_{I\setminus\{i\}})) = u^*_i(f(u'_i, u''_{I\setminus\{i\}})) \text{ for each } u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}. \tag{17}$$

Because $K \equiv \{j\}$ and $K \setminus \{j\} = \emptyset$, by (16), we find that

$$u_j(f(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) = u_j(f(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})). \tag{18}$$

Because $K \equiv \{j\}$, by (17), (18), and the **1-rectangular property**, we find that $f(u^*_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$. Because $K \equiv \{j\}$ and $u^*_i \equiv u'_i$, this implies that $f(u_j, u'_{I\setminus\{j\}}) = f(u'_j, u'_{I\setminus\{j\}})$. □

As stated above, Proposition 2 in Subsection 4.3 shows that the n-rectangular property is equivalent to the rectangular property by an argument similar to Proposition 1 in Subsection 4.3. Together with Proposition 2 of Saijo, Sjöström, and Yamato (2007), this implies that the n-rectangular property is in general stronger than strong non-bossiness which is equivalent to the 1-rectangular property by Proposition 1.

Proposition 2. *The social choice function f satisfies the* ***n-rectangular property*** *if and only if f satisfies the* ***rectangular property****.*

Proof. This proof consists of the following two claims.

Claim 1. *f satisfies the* ***n-rectangular property*** *if f satisfies the* ***rectangular property****.*

Let $u, u' \in U$, $K \subseteq I$ with $|K| \leq n$, and $i \in I \setminus K$ be such that (i) $u_i(f(u_i, u''_{I\setminus\{i\}})) = u_i(f(u'_i, u''_{I\setminus\{i\}}))$ for each $u''_{I\setminus\{i\}} \in U_{I\setminus\{i\}}$ and (ii) $u_j(f(u_j, u'_{K\setminus\{j\}}, u'_{I\setminus K})) = u_j(f(u'_j, u'_{K\setminus\{j\}}, u'_{I\setminus K}))$ for each $j \in K$. We prove this claim according to the number of $|K|$.

If $|K| = 0$, then $K = \emptyset$ and $I \setminus K = I$. This implies that Condition (ii) is redundant and it is sufficient to show that $f(u_i, u'_{I\setminus\{i\}}) = f(u'_i, u'_{I\setminus\{i\}})$ for $i \in I$ on the basis of Condition (i). Let $u''_{I\setminus\{i\}} = u'_{I\setminus\{i\}}$. Together with Condition (i), this implies that $u_i(f(u_i, u'_{I\setminus\{i\}})) = u_i(f(u'_i, u'_{I\setminus\{i\}}))$. Together with **strong non-bossiness**, this implies that $f(u_i, u'_{I\setminus\{i\}}) = f(u'_i, u'_{I\setminus\{i\}})$ because the rectangular property implies strong non-bossiness by Proposition 2 of Saijo, Sjöström, and Yamato (2007).

If $|K| = n$, then $K = I$ and $I \setminus K = \emptyset$. This implies that Condition (i) is redundant and Condition (ii) can be rewritten as follows: $u_j(f(u_j, u'_{I\setminus\{j\}})) = u_j(f(u'_j, u'_{I\setminus\{j\}}))$ for each $j \in I$. In addition, it is sufficient to show that $f(u) = f(u')$ on the basis of Condition (ii). Together with the **rectangular property**, Condition (ii) implies that $f(u) = f(u')$.

If $1 \leq |K| \leq n-1$, then $K \neq \emptyset$ and $I \setminus K \neq \emptyset$. Let $u^* \equiv (u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$. Together with Condition (ii), this implies that $u^*_j(f(u^*_j, u'_{I\setminus\{j\}})) = u^*_j(f(u'_j, u'_{I\setminus\{j\}}))$ for each $j \in I$. [21] Together with the **rectangular property**, this implies that $f(u^*) = f(u')$, that is,

$$f(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}). \tag{19}$$

Let $u''_{I\setminus\{i\}} = (u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$. Together with Condition (i), this implies that $u_i(f(u_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})) = u_i(f(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}))$. Together with **strong non-bossiness**, this implies that

$$f(u_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) \tag{20}$$

because the rectangular property implies strong non-bossiness by Proposition 2 of Saijo, Sjöström, and Yamato (2007). By (19) and (20), we find that $f(u_i, u_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}}) = f(u'_i, u'_{K\setminus\{i\}}, u'_{I\setminus\{K\cup\{i\}\}})$.

[21] Note that $u^*_j = u'_j$ for each $j \in I \setminus K$ by definition.

Claim 2. *f satisfies the **n-rectangular property** only if f satisfies the **rectangular property**.*

Let $u,u' \in U$ be such that $u_j(f(u_j,u'_{I\setminus\{j\}})) = u_j(f(u'_j,u'_{I\setminus\{j\}}))$ for each $j \in I$. Let $K \equiv I$. Together with the ***n*-rectangular property**, these imply $f(u) = f(u')$. □

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