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Farsighted Stability in a Patent Licensing Game: An Abstract Game Approach*

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Abstract

This paper deals with an abstract game which models patent licensing negotiations. We analyze a negotiation process among an external patent holder and firms, assuming that they are all farsighted. We provide a complete characterization of the symmetric farsighted stable sets, in which the payment to the patent holder is the same among licensee firms at each outcome. Given a net profit of each licensee firm, a set of outcomes is a symmetric farsighted stable set if and only if at any outcome in the set, each licensee firm receives the net profit and the number of licensee firms maximizes the patent holder's profit provided that licensee firms obtain the net profits. Although the sufficiency is shown under a remarkably mild condition, we need an additional condition for the necessity that is frequently assumed in the literature. The existence of a symmetric farsighted stable set directly follows from our result.

Keywords: symmetric farsighted stable set, patent licensing, abstract game

JEL Classification: C62, C71, D45

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1 Introduction

This paper considers farsightedly stable outcomes in negotiations on prices of information about patented technologies under a situation where the seller of the information (patent holder) has no production facility, the buyers (licensee firms) enjoy an advantage, whereas the non-buyers (non-licensee firms) suffer from a disadvantage in the market competition. Do there exist farsightedly stable outcomes in those negotiations? Can we characterize the number of licensees that maximizes the profit of the patent holders?

Licensing agreements are contract terms signed by sellers and buyers of information that result from negotiations. From this viewpoint, some researchers recently took cooperative game approaches to the analysis of the licensing negotiations.[1] A cooperative game approach to patent licensing was initiated by Tauman and Watanabe (2007), where the payoff for the patent holder in the grand coalition was focused on assuming that the number of firms competing in the market tends to infinity. Another cooperative game approach was developed by Watanabe and Muto (2008), which is explained as follows. Suppose that there are an arbitrarily finite number of firms competing in the market. In the first stage, a patent holder invites a group of firms to license negotiations. In the second stage, the patent holder and the invited firms negotiate on the fees for the patented technology. In the third stage, the licensee firms and non-licensee firms compete in a market, provided that the results of the first and second stages are commonly known to all the firms.[2]

In this model, a patent holder and firms make their decisions in the first and second stages with foreseen (re)actions of firms in the market competition in the third stage. It is thus natural to introduce a situation where a patent holder and firms also foresee reactions of others in patent licensing negotiations.[3] In order to formulate license negotiations with such a patent holder and firms who are all farsighted, we combine the first and second stages and call it a negotiation stage. We will formulate the negotiation

[1]The patent licensing problem has also been analyzed with non-cooperative games since the seminal papers by Kamien and Tauman (1984, 1986). For this strand of research, see Sen and Tauman (2007), Fan et al. (2016) and references therein.

[2]Many solution concepts in cooperative game theory have been investigated with this model. Watanabe and Muto (2008) investigated the core and the bargaining set; Kishimoto et al. (2011) investigated the Shapley value; Kishimoto and Watanabe (2017) investigated the kernel and nucleolus; Hirai and Watanbe (2018) investigated a von Neumann-Morgenstern stable set. Kishimoto (2013) extended this model to a game with non-transferable utility in order to analyze fees and royalties.

[3]This is an analogy of an indication of Diamantoudi (2005) for a cartel formation model.

stage by an abstract game due to Chwe (1994). In this negotiation stage, a patent holder and a group of firms can sign a contract on patent licensing and fees if the decision is unanimous. Given a contract of fees for the patented technology, a patent holder and a group of firms can replace the contract with another one, or the contract can be canceled unilaterally by a patent holder or one of the firms involved in the current contract. After renewing or canceling a contract, another renewal or cancellation of a contract may follow, and so on. Thus, a patent holder and firms make these decisions on contracts with subsequent renewals and cancellations of contracts.

A farsighted stable set is an appropriate solution concept for such negotiations, which is a modification of a vNM stable set (von Neumann and Morgenstern, 1944) by incorporating farsightedness of players. The idea of a farsighted stable set was first introduced by Harsanyi (1974) for coalitional games. Later, Chwe (1994) formulated a farsighted stable set for a class of abstract games that includes the negotiation stage.[4]

We completely characterize symmetric farsighted stable sets under certain conditions. In the negotiation stage, an outcome is said to be symmetric if the payoffs of the licensee firms are identical at the outcome. By the symmetry of firms, a symmetric outcome is yielded by an identical fee for patent licensing. In particular, we investigate a symmetric farsighted stable set that is a farsighted stable set consisting of symmetric outcomes. We offer a complete characterization of symmetric farsighted stable sets. To this end, a concept of a positive net profit of licensee firms is important, which is sufficiently small so that the profit of a patent holder is not fully exploited.

We show that under a remarkably mild condition, a set of symmetric outcomes is a symmetric farsighted stable set if the set satisfies the following condition for a given sufficiently small positive net profit: For any outcome in the set, (i) licensee firms uniformly enjoy the given net profit, and (ii) the number of licensee firms maximizes a patent holder's profit, provided that the payoffs of licensee firms are determined as in (i). By the finiteness of firms, this result also implies the existence of a symmetric farsighted stable set. We also show that there is no other symmetric farsighted stable set if a standard condition on the profits of non-licensee firms in the market competition stage is additionally assumed.

Our new model of licensing negotiations and a farsighted stable set supports a remarkably different manner of profit sharing from the previously reviewed literature. In previous studies, a patent holder determines the number of licensee firms to maximize

[4]Following Harsanyi (1974) and Chwe (1994), there has been a large literature on solution concepts that are based on farsighted players. Ray and Vohra (2014) give an insightful survey on the literature.

own profit ex ante with the foreseen result in the subsequent negotiations, which depends on the number of licensee firms. A farsighted stable set in our model offers another way to determine a patent holder's profit. A net profit of licensee firms is exogenously determined first, which might be regarded as an established order (of society) in the sense of von Neumann and Morgenstern (1944). Then, the number of licensee firms is determined as if a patent holder maximizes its own profit ex post provided that licensee firms obtain the given net profit.

We also investigate the relationship between farsighted stable sets and the core. In a variety of models, a farsighted stable set consisting of outcomes that yield a single payoff vector has been well investigated and showed a close relationship with the core.[5] In contrast, the union of farsighted stable sets yielding single payoff vectors may be larger than the core in our model. We also show that the close relationship is recovered if we only consider symmetric farsighted stable sets.

The remainder of the paper is organized as follows. In the next section, we formally define a model of patent licensing. In section 3, we introduce a (symmetric) farsighted stable set. The main results are stated and proved in section 4. In the final section, we conclude with some remarks.

2 A model of patent licensing

A patented technology is held by an agent called a patent holder denoted by 0. We assume that a patent holder is external in the sense that it has no production technology. Therefore, the profit of a patent holder is 0 unless it obtains fees through patent licensing. Let $N = \{1, ..., n\}$ be the set of identical firms, where $2 \leq n < \infty$, who are potential users of the patented technology held by 0. These firms compete in an oligopoly market. We call all of a patent holder and firms together players. Thus, $\{0\} \cup N$ is the set of players. A nonempty subset of $\{0\} \cup N$ is called a coalition. Let 2^N denote the set of subsets of firms including $\emptyset$, *i.e.* the power set of N. Of course, any element in 2^N except for $\emptyset$ is a coalition. Throughout this paper, we denote coalitions by capital letters and their cardinalities by the corresponding lower cases. For example, the cardinalities of coalitions $S, T, \bar{S}, S'$, and Q^h are denoted by $s, t, \bar{s}, s'$, and q^h, respectively.

This game consists of the following two stages. In the first stage, players negotiate on patent licensing and determine (i) licensee firms and (ii) fees that the licensee firms

[5] See for example, Mouleon, et al. (2011), Ray and Vohra (2015), Chander (2015), Hirai (2017), and references therein.

pay to a patent holder. We call this stage the negotiation stage. In the second stage, firms compete with each other in a market, provided that the outcome of the negotiation stage is common knowledge to all the firms. We call this stage the market competition stage. We assume that a cartel is completely prohibited in the market competition stage. Since firms are identical, the payoff of a firm resulting from the market depends only on whether it is licensed and the number of firms licensed in the negotiation stage. When $s(=0,...,n)$ firms are licensed, let $W(s)$ denote the payoff of a licensee firm and $L(s)$ denote the payoff of a non-licensee firm at the market competition stage. For notational convenience, $W(0)$ and $L(n)$ are assumed 0. Throughout the paper, the following assumption is imposed.

Assumption 1 *(i) $W(s) > L(0)$ for all $s = 1,...,n$; (ii) $L(0) > L(s) \geq 0$ for all $s = 1,...,n-1$.*

This assumption implies that the patented technology is advantageous for licensee firms and disadvantageous for non-licensee firms when it is licensed.

Our purpose is to analyze stable outcomes at the negotiation stage, where players negotiate with the foreseen payoffs (profits) that will be obtained at the subsequent market competition stage. Players negotiate to agree on a contract that determines which firms are licensed and the fees licensee firms pay. We allow asymmetric fees; The licensee firms may pay different fees to the patent holder. On the other hand, we assume that non-licensee firms pay nothing. We assume that any contract is multilateral in the sense that a contract is broken up unless all of the patent holder and licensed firms agree on it.

Contracts represent outcomes of the negotiation stage. An outcome of the negotiation stage is a pair of a set of licensee firms and a payoff allocation where there are transfers between a patent holder and licensee firms. Formally, for a given $S \in 2^N$, let

$$X^S = \left\{ x = (x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1} \,\middle|\, x_0 + \sum_{i \in S} x_i = sW(s),\ x_j = L(s) \text{ for all } j \in N \setminus S \right\}$$

denote the set of feasible payoff allocations when firms in S are licensed. Note that $X^{\emptyset} = \{(0, L(0), ..., L(0))\}$. Denote $x^{\emptyset} = (0, L(0), ..., L(0))$, which will frequently appear in later proofs. Let

$$X = \bigcup_{S \in 2^N} \left(\{S\} \times X^S \right)$$

denote the set of outcomes. Let $\bar{X} = \{(S, x) \in X | S \in 2^N, x_i = x_j, \forall i, j \in S\}$ be the set of symmetric outcomes. By the symmetry of firms, licensee firms are paying a uniform fee at an outcome if and only if the outcome is symmetric.

We turn to the definition of an effectiveness relation on X that describes rules of the negotiation. An effectiveness relation determines coalitions that are able to induce an outcome from an outcome. We denote $(S, x) \rightarrow_T (S', x')$ when a coalition T can induce $(S', x') \in X$ from $(S, x) \in X$. Since we are assuming that contracts in the negotiation are multilateral, we impose the following assumptions on effectiveness relation.

Assumption 2 *(i) For any $(S, x) \in X$, $(S, x) \rightarrow_T (\emptyset, x^{\emptyset})$ if and only if $\emptyset \neq T \subseteq \{0\} \cup S$; (ii) For any $(S, x), (S', x') \in X$ with $S' \neq \emptyset$, $(S, x) \rightarrow_T (S', x')$ if and only if $T = \{0\} \cup S'$.*

Assumption 2 requires that agreements from all members are necessary (i) to maintain a contract or (ii) to make a new contract. In (i), when a coalition $T \subseteq S$ deviates from (S, x), the patent holder and the residual firms $\{0\} \cup (S \setminus T)$ are not allowed to keep the licensing contract between only them. This contrasts with the effectivity considered in some literature of farsighted stability, for example the hedonic games considered by Diamantoudi and Xue (2003) and general patition function games considered by Chander (2015), among others. Rather, we inherit the nature of the negotiation process in the cooperative patent licensing game since Watanabe and Muto (2008). In (ii), it is legitimate that T can break up (S, x) before making (S', x') because $0 \in T = \{0\} \cup S'$ in this case, even if $S \neq \emptyset$. Note that (ii) reduces to a redistribution when $S = S' \neq \emptyset$, *i.e.* for any nonempty $S \subseteq \{0\} \cup N$ and $x, x' \in X^S$, $(S, x) \rightarrow_{\{0\} \cup S} (S, x')$.

Note that our model is essentially an abstract game due to Chwe (1994).[6]

3 Farsighted stable set

In the negotiation stage, we are assuming that the players make their decisions with the foreseen the payoffs obtained at the subsequent market competition stage. In this sense, it is implicitly assumed that the players are farsighted. Therefore, it seems consistent to consider a stability notion for farsighted players.

We employ the farsighted stable set that satisfies the stability notions *à la* von Neumann and Morgenstern (1944), where those stability notions are defined according

[6]An abstract game is originally defined as a quadruple of a set of players, a set of outcome, a profile of each player's preferences relation on the set of outcomes, and a profile of effectiveness relation. In our model, preferences relations are omitted because outcomes directly represent payoffs.

to indirect dominance relations.

We first introduce the definition of an indirect dominance relation.

Definition 1 *Let* $(S, x), (S', x') \in X$*. We say that* (S', x') *indirectly dominates* (S, x)*, which is denoted by* $(S', x') \succ (S, x)$*, if and only if there exists a sequence of outcomes* $(S^0, x^0), ..., (S^m, x^m)$ *and a sequence of coalitions* $T^1, ..., T^m$ *such that* $(S^0, x^0) = (S, x)$*,* $(S^m, x^m) = (S', x')$*, and for all* $h = 1, ..., m$*,*

- $(S^{h-1}, x^{h-1}) \rightarrow_{T^h} (S^h, x^h)$*;*
- $x'_i > x_i^{h-1}$ *for all* $i \in T^h$*.*

For simplicity, we sometimes denote the sequences of outcomes and coalitions yielding an indirect dominance relation $(S', x') \succ (S, x)$ as the following paths.

$$(S, x) = (S^0, x^0) \rightarrow_{T^1} (S^1, x^1) \rightarrow_{T^2} \cdots \rightarrow_{T^m} (S^m, x^m) = (S', x').$$

Then, a farsighted stable set is defined as follows. We also define a symmetric farsighted stable set.

Definition 2

- *We say that* $K \subseteq X$ *is a farsighted stable set iff the following two stabilities are satisfied.*

  Internal stability: *for any* $(S, x), (S', x') \in K$*,* $(S', x') \succ (S, x)$ *does not hold.*

  External stability: *for any* $(S, x) \in X \setminus K$*, there exists some* $(S', x') \in K$ *such that* $(S', x') \succ (S, x)$*.*

- *We say that* $K \subseteq X$ *is a symmetric farsighted stable set iff* K *is a farsighted stable set and* $K \subseteq \bar{X}$*.*

We state and prove two lemmas on properties of an indirect dominance relation that will be useful in a later section.

Lemma 1 *For any* $(S, x) \in \bar{X}$*,* $(S, x) \succ (\emptyset, x^{\emptyset})$ *if and only if* $x_0 > 0$ *and* $x_i > L(0)$ *for all* $i \in S$*.*

Proof. The sufficiency is straightforward since $(\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup S} (S, x)$ yields $(S, x) \succ (\emptyset, x^{\emptyset})$ by $x_0 > 0 = x_0^{\emptyset}$ and $x_i > L(0) = x_i^{\emptyset}$ for all $i \in S$.

We turn to the necessity. Fix an arbitrary $(S, x) \in \bar{X}$ such that $(S, x) \succ (\emptyset, x^{\emptyset})$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$

such that $(Q^0, z^0) = (\emptyset, x^\emptyset)$, $(Q^m, z^m) = (S, x)$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \to_{R^h} (Q^h, z^h)$ and $x_i > z_i^{h-1}$ for all $i \in R^h$. Obviously, $(S, x) \neq (\emptyset, x^\emptyset)$. Thus, $S \neq \emptyset$.

Suppose that either $x_0 \leq 0$ or $x_i \leq L(0)$ for some $i \in S$. The latter condition implies that $x_i \leq L(0)$ for all $i \in S$ by the symmetry of (S, x). If $x_0 \leq 0 = x_0^\emptyset$, then $R^1 \subseteq S$. If $x_i \leq L(0) = x_i^\emptyset$ for all $i \in S$, then $R^1 = \{0\}$ since $x_i^\emptyset = L(0) > L(s) = x_i$ for all $i \in N \setminus S$. In either case, $(Q^1, z^1) = (\emptyset, x^\emptyset)$.

Fix an arbitrary $h = 2, ..., m$. Assume that $(Q^{h-1}, z^{h-1}) = (\emptyset, x^\emptyset)$. If $x_0 \leq 0 = x_0^\emptyset$, then $R^h \subseteq S$. If $x_i \leq L(0) = x_i^\emptyset$ for all $i \in S$, then $R^h = \{0\}$ since $x_i^\emptyset = L(0) > L(s) = x_i$ for all $i \in N \setminus S$. Thus, $(Q^h, z^h) = (\emptyset, x^\emptyset)$. We eventually have $(Q^m, z^m) = (\emptyset, x^\emptyset)$, contradicting that $(Q^m, z^m) = (S, x)$, where $S \neq \emptyset$. Hence $x_0 > 0$ and $x_i > L(0)$ for all $i \in S$. ■

Note that this lemma is not retained with asymmetric outcomes. By employing the necessity of Lemma 1, we can prove the following lemma.

Lemma 2 *Let $(S, x), (T, y) \in \bar{X} \setminus \{(\emptyset, x^\emptyset)\}$ such that $(T, y) \succ (S, x)$ and either $y_0 \leq 0$ or $y_i \leq L(0)$ for all $i \in T$. Let $(Q^0, z^0), ..., (Q^m, z^m)$ and $R^1, ..., R^m$ be sequences yielding $(T, y) \succ (S, x)$, i.e. $(Q^0, z^0) = (S, x)$, $(Q^m, z^m) = (T, y)$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \to_{R^h} (Q^h, z^h)$ and $y_i > z_i^{h-1}$ for all $i \in R^h$. Then, $(Q^h, z^h) \neq (\emptyset, x^\emptyset)$ for all $h = 0, ..., m$.*

Proof. Let $(S, x), (T, y) \in \bar{X} \setminus \{(\emptyset, x^\emptyset)\}$ such that $(T, y) \succ (S, x)$ and either $y_0 \leq 0$ or $y_i \leq L(0)$ for all $i \in T$. Suppose that there exists some $\ell = 0, ..., m$ such that $(Q^\ell, z^\ell) = (\emptyset, x^\emptyset)$. Note that $0 < \ell < m$ by $(S, x) \neq (\emptyset, x^\emptyset) \neq (T, y)$. Then,

$$(Q^\ell, z^\ell) \to_{R^{\ell+1}} (Q^{\ell+1}, z^{\ell+1}) \to_{R^{\ell+2}} \cdots \to_{R^m} (Q^m, z^m)$$

yield $(T, y) = (Q^m, z^m) \succ (Q^\ell, z^\ell) = (\emptyset, x^\emptyset)$ since for all $h = \ell+1, ..., m$, $(Q^{h-1}, z^{h-1}) \to_{R^h} (Q^h, z^h)$ and $y_i > z_i^{h-1}$ for all $i \in R^h$. This contradicts the necessity of Lemma 1 by the choice of (T, y). Hence, $(Q^h, z^h) \neq (\emptyset, x^\emptyset)$ for all $h = 0, ..., m$. ■

4 Main results

We state and prove a characterization of symmetric farsighted stable sets of the negotiation stage that is the main result of this paper. We begin with preparations. Let

$$E = \{\varepsilon \in \mathbb{R}_{++} \,|\, s(W(s) - L(0) - \varepsilon) > 0 \text{ for some } s = 1, ..., n\}.$$

For any $\varepsilon \in E$, define

$$B(\varepsilon) = \arg\max_{s=1,...,n} s(W(s) - L(0) - \varepsilon).$$

Note that E is the set of net profits of licensee firms where a patent holder and licensee firms are made better off than $x^{\emptyset}$. Therefore, E can be regarded as the set of strictly individually rational net profits of licensee firms since any of a patent holder and licensee firms can solely induce $(\emptyset, x^{\emptyset})$. Given $\varepsilon \in E$, $B(\varepsilon)$ is the set of the optimal numbers of licensee firms for a patent holder, provided that each licensee firm obtains net profit ε. Note that $E \neq \emptyset$ by Assumption 1(i). Note also that $B(\varepsilon) \neq \emptyset$ for all $\varepsilon \in E$ by the finiteness of the firms.

For any $\varepsilon \in E$, define

$$\bar{X}(\varepsilon) = \{(S, x) \in \bar{X} | s \in B(\varepsilon), x_0 = s(W(s) - L(0) - \varepsilon), x_i = L(0) + \varepsilon \text{ for all } i \in S\}.$$

In $\bar{X}(\varepsilon)$, each licensee firm receives an identical net profit ε, which is exogenously given. Such outcomes are yielded by a uniform fee for patent licensing since firms are symmetric. In $\bar{X}(\varepsilon)$, a patent holder determines the number of licensee firms as if it maximizes own profit subject to the given uniform fee. In this sense, a patent holder behaves like a price (fee)-taker in a symmetric farsighted stable set. Note that $B(\varepsilon)$ may include two or more natural numbers for a given $\varepsilon \in E$. Thus, $\bar{X}(\varepsilon)$ may include outcomes (S, x) and (S', x') such that $s \neq s'$. Anyway, we have $x_0 = x'_0$ and $x_i = x'_j$ for all $i \in S$ and all $j \in S'$ by the definitions of $B(\varepsilon)$ and $\bar{X}(\varepsilon)$.

Now, we turn to our main results.

Theorem 1 *For any $\varepsilon \in E$, $\bar{X}(\varepsilon)$ is a symmetric farsighted stable set.*

Proof. Fix an arbitrary $\varepsilon^* \in E$. Let $S^* \in 2^N$ with $S^* \neq \emptyset$ such that $s^* \in B(\varepsilon^*)$. We first show internal stability. Fix arbitrary $(S, x), (S', x') \in \bar{X}(\varepsilon^*)$. Suppose that $(S', x') \succ (S, x)$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (S, x)$, $(Q^m, z^m) = (S', x')$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $x'_i > z_i^{h-1}$ for all $i \in R^h$. By $x_0 = x'_0 = s^*(W(s^*) - L(0) - \varepsilon^*)$, $0 \notin R^1$. Then, $R^1 \subseteq S$ by the definition of the effectiveness relation. However, for all $i \in S$, $x_i = L(0) + \varepsilon^* \geq x'_i$ because $x'_i = L(0) + \varepsilon^*$ when $i \in S'$ and $x'_i = L(s') < L(0)$ when $i \notin S'$. Thus, $R^1 = \emptyset$, contradicting that R^1 is a coalition. Hence, $\bar{X}(\varepsilon^*)$ is internally stable.

Next, we show external stability. Fix an arbitrary $(T, y) \in X \setminus \bar{X}(\varepsilon^*)$. If $(T, y) = (\emptyset, x^{\emptyset})$, then $(S, x) \succ (T, y)$ for any $(S, x) \in \bar{X}(\varepsilon^*)$ by Lemma 1. Thus, assume that $T \neq \emptyset$. We distinguish two cases.

Case 1. $y_0 < s^*(W(s^*) - L(0) - \varepsilon^*)$.

Let (S^*, x^*) be a symmetric outcome such that $x_0^* = s^*(W(s^*) - L(0) - \varepsilon^*)$, $x_i^* = L(0) + \varepsilon^*$ for all $i \in S^*$, and $x_i^* = L(s^*)$ for all $i \in N \setminus S^*$. Note that $(S^*, x^*) \in \bar{X}(\varepsilon^*)$. Then,

$$(T, y) \rightarrow_{\{0\}} (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup S^*} (S^*, x^*),$$

yield $(S^*, x^*) \succ (T, y)$ by

- $x_0^* = s^*(W(s^*) - L(0) - \varepsilon^*) > \max\{0, y_0\} = \max\{x_0^{\emptyset}, y_0\}$ and
- $x_i^* = L(0) + \varepsilon^* > L(0) = x_i^{\emptyset}$ for all $i \in S^*$.

Case 2. $y_0 \geq s^*(W(s^*) - L(0) - \varepsilon^*)$.

We claim that $y_i < L(0) + \varepsilon^*$ for some $i \in T$. Suppose that $y_i \geq L(0) + \varepsilon^*$ for all $i \in T$. Then,

$$tW(t) = \sum_{i \in \{0\} \cup T} y_i \geq s^*(W(s^*) - L(0) - \varepsilon^*) + t(L(0) + \varepsilon^*), \tag{1}$$

which is equivalent to

$$t(W(t) - L(0) - \varepsilon^*) \geq s^*(W(s^*) - L(0) - \varepsilon^*). \tag{2}$$

If $y_j > L(0) + \varepsilon^*$ for some $j \in T$, then (1) as well as (2) holds in a strict inequality, contradicting $s^* \in B(\varepsilon^*)$. Thus, $y_i = L(0) + \varepsilon^*$ for all $i \in T$. By $tW(t) = \sum_{i \in \{0\} \cup T} y_i$, $y_0 = t(W(t) - L(0) - \varepsilon^*)$. Then, $t \in B(\varepsilon^*)$ by (2) and $s^* \in B(\varepsilon^*)$. Therefore, $(T, y) \in \bar{X}(\varepsilon^*)$, contradicting $(T, y) \in X \setminus \bar{X}(\varepsilon^*)$. Hence, there exists some $j \in T$ such that $y_j < L(0) + \varepsilon^*$.

Let $(S', x') \in \bar{X}$ such that $j \in S'$, $s' \in B(\varepsilon^*)$, $x_0' = s'(W(s') - L(0) - \varepsilon^*)$, $x_i' = L(0) + \varepsilon^*$ for all $i \in S'$, and $x_i' = L(s')$ for all $i \in N \setminus S'$. Note that $(S', x') \in \bar{X}(\varepsilon^*)$. Then,

$$(T, y) \rightarrow_{\{j\}} (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup S'} (S', x').$$

yield $(S', x') \succ (T, y)$ by

- $x_j' = L(0) + \varepsilon^* > y_j$,

- $x'_i = L(0) + \varepsilon^* > x_i^{\emptyset}$ for all $i \in S'$, and
- $x'_0 = s^*(W(s^*) - L(0) - \varepsilon^*) > 0 = x_0^{\emptyset}$.

Hence $\bar{X}(\varepsilon^*)$ is externally stable. ■

The following corollary is immediate from Theorem 1, $E \neq \emptyset$, and $B(\varepsilon) \neq \emptyset$ for any $\varepsilon \in E$.

Corollary 1 *A symmetric farsighted stable set exists.*

To obtain the inverse relationship, we assume an additional condition such that $L(s)$ is nonincreasing in $s = 1, ..., n-1$. This condition is satisfied when the patented technology is a cost reduction technology and the market competition stage is a Cournot oligopoly with certain conditions. See Watanabe and Muto (2008) and Kishimoto et al. (2011). Moreover, this condition has been assumed in the literature, *e.g.* Kishimoto et al. (2011) and Hirai and Watanbe (2018).

Theorem 2 *Assume additionally that $L(s)$ is nonincreasing in $s = 1, ..., n-1$. If $\bar{K}$ is a symmetric farsighted stable set, then $\bar{K} = \bar{X}(\varepsilon)$ for some $\varepsilon \in E$.*

Proof. Fix an arbitrary symmetric farsighted stable set $\bar{K}$.

Claim 1 $(\emptyset, x^{\emptyset}) \notin \bar{K}$.

Proof of Claim 1. Suppose that $(\emptyset, x^{\emptyset}) \in \bar{K}$. Let $(N, \hat{x}) \in X$ such that $\hat{x}_0 = n(W(n) - L(0) - \delta)$ and $\hat{x}_i = L(0) + \delta$ for all $i \in N$, where $\delta > 0$ is a sufficiently small real number so that $\hat{x}_0 > 0$. Note that we can take such $(N, \hat{x})$ by $W(n) > L(0)$. Then, $(N, \hat{x}) \succ (\emptyset, x^{\emptyset})$ by Lemma 1. Thus, $(N, \hat{x}) \notin \bar{K}$ by internal stability of $\bar{K}$. It is also easy to see that $(\emptyset, x^{\emptyset}) \succ (N, \hat{x})$ is impossible by $\hat{x}_i > x_i^{\emptyset}$ for all $i \in \{0\} \cup N$. Thus, there exists some $(S, x) \in \bar{K}$ such that $S \neq \emptyset$ and $(S, x) \succ (N, \hat{x})$ by external stability of $\bar{K}$.

If $x_i < x_i^{\emptyset}$ for some $i \in \{0\} \cup S$, then $(\emptyset, x^{\emptyset}) \succ (S, x)$ holds by $(S, x) \rightarrow_{\{i\}} (\emptyset, x^{\emptyset})$, contradicting internal stability of $\bar{K}$. Thus, $x_0 \geq 0$ and $x_i \geq L(0)$ for all $i \in S$. Moreover, by internal stability of $\bar{K}$ and the sufficiency of Lemma 1, either $x_0 = 0$ or $x_i = L(0)$ for all $i \in S$. Note that $x_i = L(s) < L(0)$ for all $i \in N \setminus S$.

By $(S, x) \succ (N, \hat{x})$, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (N, \hat{x})$, $(Q^m, z^m) = (S, x)$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $x_i > z_i^{h-1}$ for all $i \in R^h$. If $x_0 = 0$, then $0 \notin R^1$

by $x_i > z_i^0 = \hat{x}_i$ for all $i \in R^1$. If $x_i = L(0)$ for all $i \in S$, then $R^1 = \{0\}$ since $\hat{x}_i > L(0) = x_i$ for all $i \in S$ and $\hat{x}_i > L(0) > L(s) = x_i$ for all $i \in N \setminus S$, while it is assumed that $x_i > z_i^0 = \hat{x}_i$ for all $i \in R^1$. Therefore, $(Q^1, z^1) = (\emptyset, x^\emptyset)$ in either case. This contradicts Lemma 2 since either $x_0 = 0$ or $x_i = L(0)$ for all $i \in S$. Hence $(\emptyset, x^\emptyset) \notin \bar{K}$. □

By Lemma 1, Claim 1, and external stability of $\bar{K}$, there exists some $(\bar{S}, \bar{x}) \in \bar{K}$ such that $\bar{S} \neq \emptyset$, $\bar{x}_0 > 0$ and $\bar{x}_i > L(0)$ for all $i \in \bar{S}$. Without loss of generality, we may choose $(\bar{S}, \bar{x})$ so that for any $(S', x') \in \bar{K}$ $x'_0 = \bar{x}_0$ implies $x'_j \leq \bar{x}_i$ for all $j \in S'$ and $i \in \bar{S}$. We can take such an outcome since $\{(T, y) \in \bar{X} | y_0 = \bar{x}_0\}$ is finite. In words, $(\bar{S}, \bar{x})$ gives the largest payoffs for the licensee firms among the outcomes in $\bar{K}$ that guarantee the same patent holder's payoffs. Denote $\bar{x}_i = L(0) + \bar{\varepsilon}$ for all $i \in \bar{S}$. Note that $\bar{\varepsilon} \in E$ by $\bar{s}(W(\bar{s}) - L(0) - \bar{\varepsilon}) = \bar{x}_0 > 0$. Throughout this proof, this $(\bar{S}, \bar{x})$ is fixed.

In what follows, we state and prove five claims. Claims 2 and 3 give necessary conditions for an outcome in $\bar{K}$ such that the patent holder's profit is different from $\bar{x}_0$. By using these conditions, we show in Claim 4 that the patent holder's profit is actually identical at any outcome in $\bar{K}$. We also show that that the licensee firms' profits are identical across outcomes in $\bar{K}$ in Claim 5. In Claim 6, we show that the number of licensee firms at any outcome in $\bar{K}$ maximizes the patent holder's profit provided that the licensee firms' profits are identically $L(0) + \bar{\varepsilon}$.

Claim 2 *For any $(T, y) \in \bar{K}$, $y_0 \neq \bar{x}_0$ implies (i) $y_0 > \bar{x}_0$, (ii) $\bar{S} \cap T = \emptyset$, and (iii) $y_i = L(\bar{s})$ for all $i \in T$.*

Proof of Claim 2. Fix an arbitrary $(T, y) \in \bar{K}$. Assume that $y_0 \neq \bar{x}_0$. Note that both $(\bar{S}, \bar{x})$ and (T, y) are symmetric. First, suppose that $y_0 < \bar{x}_0$. Then, $(T, y) \rightarrow_{\{0\}} (\emptyset, x^\emptyset) \rightarrow_{\{0\} \cup \bar{S}} (\bar{S}, \bar{x})$ yield $(\bar{S}, \bar{x}) \succ (T, y)$, contradicting internal stability of $\bar{K}$. Hence, we obtain (i) $y_0 > \bar{x}_0$.

Second, suppose that $\bar{S} \cap T \neq \emptyset$. If $\bar{x}_i > y_i$ for all $i \in \bar{S} \cap T$, then $(T, y) \rightarrow_{\bar{S} \cap T} (\emptyset, x^\emptyset) \rightarrow_{\{0\} \cup \bar{S}} (\bar{S}, \bar{x})$ yield $(\bar{S}, \bar{x}) \succ (T, y)$ by $\bar{x}_0 > 0 = x_0^\emptyset$ and $\bar{x}_i > L(0) = x_i^\emptyset$ for all $i \in \bar{S}$. This contradicts internal stability of $\bar{K}$. Assume, therefore, that $\bar{x}_i \leq y_i$ for all $i \in \bar{S} \cap T$. Then, $y_i \geq \bar{x}_i > L(0)$ for all $i \in \bar{S} \cap T$. By the symmetry of (T, y), $\bar{y}_i > L(0)$ for all $i \in T$. Thus, $(\bar{S}, \bar{x}) \rightarrow_{\{0\}} (\emptyset, x^\emptyset) \rightarrow_{\{0\} \cup T} (T, y)$ yield $(T, y) \succ (\bar{S}, \bar{x})$ by $y_0 > \bar{x}_0 > 0 = x_0^\emptyset$, contradicting internal stability of $\bar{K}$. Hence, we have (ii) $\bar{S} \cap T = \emptyset$.

Finally, we show (iii). If $y_i > L(\bar{s})$ for all $i \in T$, then $(\bar{S}, \bar{x}) \rightarrow_{\{0\} \cup T} (T, y)$ yields $(T, y) \succ (\bar{S}, \bar{x})$ by

- $y_0 > \bar{x}_0$ and
- $\bar{x}_i = L(\bar{s}) < y_i$ for all $i \in T$,

where the latter follows from (ii) $\bar{S} \cap T = \emptyset$. If $y_i < L(\bar{s})$ for all $i \in T$, then $(T, y) \rightarrow_T (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup \bar{S}} (\bar{S}, \bar{x})$ yield $(\bar{S}, \bar{x}) \succ (T, y)$ by

- $\bar{x}_0 > 0 = x_0^{\emptyset}$,
- $\bar{x}_i > L(0) = x_i^{\emptyset}$ for all $i \in \bar{S}$, and
- $\bar{x}_i = L(\bar{s}) > y_i$ for all $i \in T$,

where the third statement follows from (ii) $\bar{S} \cap T = \emptyset$. Either case contradicts internal stability of $\bar{K}$. Hence, we have (iii) $y_i = L(\bar{s})$ for all $i \in T$. □

Claim 3 *For any $(T, y) \in \bar{K}$, $y_0 \neq \bar{x}_0$ implies $L(t) \leq L(\bar{s})$.*

Proof of Claim 3. Let $(T, y) \in \bar{K}$ such that $y_0 \neq \bar{x}_0$. Note that $y_0 > \bar{x}_0$, $\bar{S} \cap T = \emptyset$, and $y_i = L(\bar{s})$ for all $i \in T$ by Claim 2. Suppose that $L(t) > L(\bar{s})$. Without loss of generality, we may assume that $L(t) \geq L(t')$ for any symmetric outcome $(T', y') \in \bar{K}$ such that $y'_0 \neq \bar{x}_0$.

Let $(T, \hat{y})$ be a symmetric outcome such that $\hat{y}_i = y_i + \hat{\varepsilon} = L(\bar{s}) + \hat{\varepsilon}$ for all $i \in T$ and $\hat{y}_i = L(t) = y_i$ for all $i \in N \setminus T$, where $\hat{\varepsilon} > 0$ is sufficiently small so that $\hat{y}_0 = y_0 - t\hat{\varepsilon} > \bar{x}_0$ and $\hat{y}_i = L(\bar{s}) + \hat{\varepsilon} < L(t) < L(0)$ for all $i \in T$. Then, $(T, \hat{y}) \notin \bar{K}$ by Claim 2(iii).

We show that neither $(\bar{S}, \bar{x}) \succ (T, \hat{y})$ nor $(T, y) \succ (T, \hat{y})$. First, suppose that $(\bar{S}, \bar{x}) \succ (T, \hat{y})$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (T, \hat{y})$, $(Q^m, z^m) = (\bar{S}, \bar{x})$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < \bar{x}_i$ for all $i \in R^h$. By $\bar{x}_0 < \hat{y}_0$, $0 \notin R^1$. Then, $R^1 \subseteq T$ by the assumption on the effectiveness relation correspondence. On the other hand, $\bar{x}_i = L(\bar{s}) < \hat{y}_i$ for all $i \in T$ by $\bar{S} \cap T = \emptyset$. Thus, $R^1 = \emptyset$, contradicting that R^1 is a coalition. Hence, $(\bar{S}, \bar{x}) \succ (T, \hat{y})$ is impossible.

Next, suppose that $(T, y) \succ (T, \hat{y})$. Then, there exists a sequence of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and a sequence of coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (T, \hat{y})$, $(Q^m, z^m) = (T, y)$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < y_i$ for all $i \in R^h$. By $\hat{y}_i \geq y_i$ for all $i \in N$ and the nonemptiness of R^1, $R^1 = \{0\}$. Then, $(Q^1, z^1) = (\emptyset, x^{\emptyset})$. This contradicts Lemma 2 by $y_i = L(\bar{s}) < L(0)$ for all $i \in T$. Hence, $(T, y) \succ (T, \hat{y})$ is also impossible.

By external stability of $\bar{K}$, there exists some $(S',x') \in \bar{K}$ such that $(S',x') \succ (T,\hat{y})$. Note that $S' \neq \emptyset$ by Claim 1. Then, there exist sequences of outcomes $(Q^0,z^0),...,(Q^m,z^m)$ and coalitions $R^1,...,R^m$ such that $(Q^0,z^0)=(T,\hat{y})$, $(Q^m,z^m)=(S',x')$, and for all $h=1,...,m$, $(Q^{h-1},z^{h-1}) \to_{R^h} (Q^h,z^h)$ and $z_i^{h-1} < x'_i$ for all $i \in R^h$.

Suppose that $x'_0 = \bar{x}_0$. By $x'_0 = \bar{x}_0 < \hat{y}_0$, $0 \notin R^1$. Then, $R^1 \subseteq T$ and $(Q^1,z^1)=(\emptyset,x^{\emptyset})$ by the assumption on the effectiveness relation. Then, $(\emptyset,x^{\emptyset}) \to_{R^2} (Q^2,z^2) \to_{R^3} \cdots \to_{R^m} (S',x')$ yield $(S',x') \succ (\emptyset,x^{\emptyset})$ by the choices of the outcomes and coalitions constituting these paths. It must be $x'_i > L(0)$ for all $i \in S'$ by Lemma 1. If both $R^1 \setminus S' \neq \emptyset$ and $L(s') \leq L(\bar{s}) + \hat{\varepsilon}$, then $x'_j = L(s') \leq L(\bar{s}) + \hat{\varepsilon} = \hat{y}_j$ for all $j \in R^1 \setminus S' \subseteq T \setminus S'$. This contradicts that $x'_i > \hat{y}_i$ for all $i \in R^1$. Thus, $R^1 \subseteq S'$ or $L(s') > L(\bar{s}) + \hat{\varepsilon}$. Assume that $R^1 \subseteq S'$. Note that $R^1 \subseteq S' \cap T$. Then, $(T,y) \to_{R^1} (\emptyset,x^{\emptyset}) \to_{\{0\}\cup S'} (S',x')$ yield $(S',x') \succ (T,y)$ by

- $x'_0 = \bar{x}_0 > 0 = x_0^{\emptyset}$,
- $x'_i > L(0) = x_i^{\emptyset}$ for all $i \in S'$, and
- $x'_i > L(0) > L(\bar{s}) = y_i$ for all $i \in R^1$, where $R^1 \subseteq S' \cap T$.

This contradicts internal stability of $\bar{K}$. Assume, therefore, that $L(s') > L(\bar{s}) + \hat{\varepsilon}$. Then,

$$x'_i = L(s') > L(\bar{s}) + \hat{\varepsilon} > L(\bar{s}) = y_i \text{ for all } i \in T \setminus S'.$$

Together with $x'_i > L(0) > L(\bar{s}) = y_i$ for all $i \in S' \cap T$, we obtain $x'_i > y_i$ for all $i \in T$. Then, $(T,y) \to_T (\emptyset,x^{\emptyset}) \to_{\{0\}\cup S'} (S',x')$ yield $(S',x') \succ (T,y)$ by $x'_0 = \bar{x}_0 > 0 = x_0^{\emptyset}$ and $x'_i > L(0) = x_i^{\emptyset}$ for all $i \in S'$. This contradicts internal stability of $\bar{K}$. Hence, $x'_0 \neq \bar{x}_0$.

By Claim 2, $x'_0 > \bar{x}_0$, $\bar{S} \cap S' = \emptyset$, and $x'_i = L(\bar{s})$ for all $i \in S'$. Note that $L(t) \geq L(s')$ since (T,y) was chosen so that $L(t) \geq L(t')$ for any $(T',y') \in \bar{K}$ with $y'_0 \neq \bar{x}_0$. If there exists some $\ell = 1,...,m$ such that $0 \notin R^{\ell}$ or $R^{\ell} = \{0\}$, then $(Q^{\ell},z^{\ell}) = (\emptyset,x^{\emptyset})$. This contradicts Lemma 2 by $x'_i < L(0)$ for all $i \in S'$. Hence,

$$0 \in R^h \text{ and } R^h \setminus \{0\} \neq \emptyset \text{ for all } h=1,...,m. \tag{3}$$

For any $i \notin \{0\} \cup T$, we have that

$$\hat{y}_i = L(t) > L(\bar{s}) = x'_i \text{ if } i \in S';$$
$$\hat{y}_i = L(t) \geq L(s') = x'_i \text{ if } i \notin S'.$$

Therefore, $R^1 \subseteq \{0\} \cup T$, and thus, $Q^1 = R^1 \setminus \{0\} \subseteq T$. Suppose that $Q^1 \cap S' \neq \emptyset$. Then, $\hat{y}_i = L(\bar{s}) + \hat{\varepsilon} > L(\bar{s}) = x'_i$ for all $i \in Q^1 \cap S'$ by $Q^1 \subseteq T$. This contradicts that $\hat{y}_i = z_i^0 < x'_i$ for all $i \in R^1$ by $\emptyset \neq Q^1 \cap S' \subseteq R^1$. Therefore, $Q^1 \cap S' = \emptyset$.

For each $h = 2, ..., m$, we claim that if $Q^{h-1} \subseteq T$, then $Q^h \subseteq Q^{h-1}$. Fix an arbitrary $\ell = 2, ..., m$. Assume that $Q^{\ell-1} \subseteq T$. For any $i \notin \{0\} \cup Q^{\ell-1}$, we have that

$$z_i^{\ell-1} = L(q^{\ell-1}) \geq L(t) > L(\bar{s}) = x'_i \text{ if } i \in S';$$
$$z_i^{\ell-1} = L(q^{\ell-1}) \geq L(t) \geq L(s') = x'_i \text{ if } i \notin S'$$

since $L(s)$ is nonincreasing in $s = 1, ..., n-1$. Therefore, $R^\ell \subseteq \{0\} \cup Q^{\ell-1}$ since $z_i^{\ell-1} < x'_i$ for all $i \in R^\ell$. Thus, $Q^\ell = R^\ell \setminus \{0\} \subseteq Q^{\ell-1}$ by (3).

Then, $Q^m \subseteq \cdots \subseteq Q^1$. By $Q^1 \cap S' = \emptyset$, $Q^h \cap S' = \emptyset$ for all $h = 1, ..., m$. This contradicts that $Q^m = S' \neq \emptyset$. Hence, $L(t) \leq L(\bar{s})$. By the choice of (T, y), $L(t') \leq L(\bar{s})$ for any $(T', y') \in \bar{K}$ such that $y'_0 \neq \bar{x}_0$. □

Claim 4 *For any* $(T, y) \in \bar{K}$, $y_0 = \bar{x}_0$

Proof of Claim 4. Suppose that there exists some $(T, y) \in \bar{K}$ such that $y_0 \neq \bar{x}_0$. Note that $T \neq \emptyset$. Note also that (T, y) is symmetric by the definition of $\bar{K}$. By Claim 2, $y_0 > \bar{x}_0$, $\bar{S} \cap T = \emptyset$, and $y_i = L(\bar{s})$ for all $i \in T$. By Claim 3, $L(t) \leq L(\bar{s})$.

Let $S^* \subseteq N$ be a coalition such that $s^* = \bar{s}$, $S^* \subseteq \bar{S} \cup T$, and $S^* \neq \bar{S}$. Note that we can take such S^* by $\bar{S} \cap T = \emptyset$ and $\bar{S}, T \neq \emptyset$. Note also that $S^* \cap T \neq \emptyset$ by the choice of S^*. Let (S^*, x^*) be a symmetric outcome such that

$$x_i^* = \begin{cases} \bar{x}_0 = \bar{s}(W(\bar{s}) - L(0) - \bar{\varepsilon}) & \text{if } i = 0; \\ L(0) + \bar{\varepsilon} & \text{if } i \in S^*; \\ L(\bar{s}) & \text{if } i \in N \setminus S^*. \end{cases}$$

Note that (S^*, x^*) satisfies the definition of an outcome by $s^* = \bar{s}$. Since either $y_i = L(\bar{s})$ or $y_i = L(t)$ for all $i \in N$, $x_i^* > L(0) > y_i$ for all $i \in S^*$ by Assumption 1(b). Then, $(T, y) \rightarrow_{S^* \cap T} (\emptyset, x^\emptyset) \rightarrow_{\{0\} \cup S^*} (S^*, x^*)$ yield $(S^*, x^*) \succ (T, y)$ by

- $x_i^* > y_i$ for all $i \in S^* \cap T$, where $S^* \cap T \neq \emptyset$,
- $x_0^* = \bar{x}_0 > 0 = x_0^\emptyset$, and
- $x_i^* > L(0) = x_i^\emptyset$ for all $i \in S^*$.

Thus, $(S^*, x^*) \notin \bar{K}$ by internal stability of $\bar{K}$.

By the symmetry and external stability of $\bar{K}$, there exists a symmetric $(S', x') \in \bar{K}$ such that $(S', x') \succ (S^*, x^*)$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (S^*, x^*)$, $(Q^m, z^m) = (S', x')$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < x'_i$ for all $i \in R^h$. By Claim 2, $x'_0 \geq \bar{x}_0$.

Suppose that $x'_0 = \bar{x}_0 = x^*_0$. Recall that $(\bar{S}, \bar{x})$ is chosen so that for any $(S'', x'') \in \bar{K}$, $\bar{x}_0 = x''_0$ implies $x''_j \leq \bar{x}_i$ for all $j \in S''$ and $i \in \bar{S}$. Then, by the definition of (S^*, x^*), $x'_0 = \bar{x}_0 = x^*_0$ implies $x'_j \leq x^*_i$ for all $j \in S'$ and $i \in S^*$. By $x'_0 = \bar{x}_0 = x^*_0$, $0 \notin R^1$, and thus, $R^1 \subseteq S^*$ by the assumption on the effectiveness relation. Therefore, $x'_i > x^*_i = L(0) + \bar{\varepsilon}$ for all $i \in R^1 \subseteq S^*$. However, for all $i \in R^1 \subseteq S^*$,

$$x'_i = L(s') < L(0) < x^*_i \text{ if } i \notin S';$$
$$x'_i \leq x^*_i \text{ if } i \in S'.$$

This is a contradiction.

Assume, therefore, that $x'_0 > \bar{x}_0$. By the choice of (S^*, x^*), $x^*_i \geq L(\bar{s})$ for all $i \in N$. On the other hand, $x'_i \leq L(\bar{s})$ for all $i \in N$ by Claim 2(iii) and Claim 3. Therefore, $R^1 = \{0\}$ and $(Q^1, z^1) = (\emptyset, x^{\emptyset})$. This contradicts Lemma 2 since we have $x'_i = L(\bar{s}) < L(0)$ for all $i \in S'$ by Claim 2(iii). Hence, $\bar{x}_0 = y_0$. □

Claim 5 *For any $(S', x') \in \bar{K}$, $x'_i = L(0) + \bar{\varepsilon}$ for all $i \in S'$.*

Proof of Claim 5. Fix an arbitrary $(S', x') \in \bar{K}$, which is symmetric. Suppose that $x'_j \neq L(0) + \bar{\varepsilon}$ for all $j \in S'$. Note that $S' \neq \emptyset$ by Lemma 1. Recall again that $(\bar{S}, \bar{x})$ is chosen so that for any $(S'', x'') \in \bar{K}$, $\bar{x}_0 = x''_0$ implies $x''_j \leq \bar{x}_i$ for all $j \in S''$ and $i \in \bar{S}$. Then, by Claim 4, $x'_j < L(0) + \bar{\varepsilon} = \bar{x}_i$ for all $i \in \bar{S}$ and $j \in S'$.

Suppose that $\bar{S} \cap S' \neq \emptyset$. Then, $(S', x') \rightarrow_{\bar{S} \cap S'} (\emptyset, x^{\emptyset}) \rightarrow_{\{0\} \cup \bar{S}} (\bar{S}, \bar{x})$ yield $(\bar{S}, \bar{x}) \succ (S', x')$ by

- $x'_i < \bar{x}_i$ for all $i \in \bar{S} \cap S' \neq \emptyset$,
- $\bar{x}_0 > 0 = x^{\emptyset}_0$, and
- $\bar{x}_i > L(0) = x^{\emptyset}_i$ for all $i \in \bar{S}$.

This contradicts internal stability of $\bar{K}$. Therefore, assume that $\bar{S} \cap S' = \emptyset$.

Let (S^*, x^*) be a symmetric outcome such that $s^* = \bar{s}$, $S' \cap S^* \neq \emptyset$, $S^* \neq \bar{S}$, and

$$x_i^* = \begin{cases} \bar{s}(W(\bar{s}) - L(0) - \bar{\varepsilon}) & \text{if } i = 0; \\ L(0) + \bar{\varepsilon} & \text{if } i \in S^*; \\ L(\bar{s}) & \text{if } i \in N \setminus S^*. \end{cases}$$

Note that (S^*, x^*) satisfies the definition of an outcome by $s^* = \bar{s}$. Note also that we can take such S^* by replacing a firm in $\bar{S}$ with a firm in S' since $\bar{S} \cap S' = \emptyset$, where $\bar{S}, S' \neq \emptyset$. Moreover, $x_0^* = \bar{x}_0$ and $x_i^* = \bar{x}_j$ for all $i \in S^*$ and $j \in \bar{S}$. Then, $(S', x') \rightarrow_{S' \cap S^*} (\emptyset, x^\emptyset) \rightarrow_{\{0\} \cup S^*} (S^*, x^*)$ yield $(S^*, x^*) \succ (S', x')$ by

- $x_i^* = L(0) + \bar{\varepsilon} > x_i'$ for all $i \in S' \cap S^*$,
- $x_0^* = \bar{x}_0 > 0 = x_0^\emptyset$, and
- $x_i^* = L(0) + \bar{\varepsilon} > L(0) = x_i^\emptyset$ for all $i \in S^*$.

Thus, $(S^*, x^*) \notin \bar{K}$ by internal stability of $\bar{K}$.

By external stability of $\bar{K}$, there exists some $(\hat{S}, \hat{x}) \in \bar{K}$ such that $(\hat{S}, \hat{x}) \succ (S^*, x^*)$. Thus, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (S^*, x^*)$, $(Q^m, z^m) = (\hat{S}, \hat{x})$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < \hat{x}_i$ for all $i \in R^h$. By Claim 4, $\hat{x}_0 = \bar{x}_0 = x_0^*$. Thus, $0 \notin R^1$ and $R^1 \subseteq S^*$. By the choice of $(\bar{S}, \bar{x})$ and the definition of (S^*, x^*),

$$\hat{x}_i \leq L(x) + \bar{\varepsilon} = x_i^* \text{ if } i \in S^* \cap \hat{S};$$
$$\hat{x}_i < L(0) < L(0) + \bar{\varepsilon} = x_i^* \text{ if } i \in S^* \setminus \hat{S}.$$

Thus, $S^* \cap R^1 = \emptyset$ that implies $R^1 = \emptyset$, contradicting that R^1 is a coalition. Hence, $x_j' = \bar{x}_i$ for all $i \in \bar{S}$ and $j \in S'$. □

Claim 6 *$\bar{s} \in B(\bar{\varepsilon})$.*

Proof of Claim 6. Suppose that $\bar{s} \notin B(\bar{\varepsilon})$. Let $(S^*, x^*) \in \bar{X}$ such that $s^* \in B(\bar{\varepsilon})$ and

$$x_i^* = \begin{cases} s^*(W(s^*) - L(0) - \bar{\varepsilon}) & \text{if } i = 0; \\ L(0) + \bar{\varepsilon} & \text{if } i \in S^*; \\ L(s^*) & \text{if } i \in N \setminus S^*. \end{cases}$$

Note that $x_0^* > \bar{x}_0$ by the definition of $B(\bar{\varepsilon})$. Then, $(S^*, x^*) \notin \bar{K}$ by Claim 4. By external stability of $\bar{K}$, there exists some $(S', x') \in \bar{K}$ such that $(S', x') \succ (S^*, x^*)$. Then, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (S^*, x^*)$, $(Q^m, z^m) = (S', x')$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \to_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < x_i'$ for all $i \in R^h$. Note that $x_0' = \bar{x}_0 < x_0^*$ by Claim 4. Thus, $0 \notin R^1$. Then, $R^1 \subseteq S^*$ by the assumption on the effectiveness relation. Note that $x_i' = L(0) + \bar{\varepsilon}$ for all $i \in S'$ by Claim 5. For any $i \in S^*$,

$$x_i^* = L(0) + \bar{\varepsilon} = x_i' \text{ if } i \in S';$$
$$x_i^* = L(0) + \bar{\varepsilon} > L(s') = x_i' \text{ if } i \notin S'.$$

Thus, $R^1 \cap S^* = \emptyset$ that implies $R^1 = \emptyset$ by $R^1 \subseteq S^*$. This contradicts that R^1 is a coalition. Hence, $\bar{s} \in B(\bar{\varepsilon})$. □

By Claim 4-6, $\bar{K} \subseteq \bar{X}(\bar{\varepsilon})$. Then, we obtain $\bar{K} = \bar{X}(\bar{\varepsilon})$ by Theorem 1, internal stability of $\bar{X}(\bar{\varepsilon})$, and external stability of $\bar{K}$. ■

Theorems 1 and 2 completely characterize symmetric farsighted stable sets when $L(s)$ is nonincreasing in s. In particular, there exists a symmetric and singleton farsighted stable set if and only if $\{n\} = B(\varepsilon)$ for some $\varepsilon \in E$. As we mentioned in Section 1, farsighted stable sets yielding single payoffs have been well investigated and shown to have close relationship with the myopic cores. In our model, any outcome in $\mathring{C}$ solely constitutes a farsighted stable set even if the outcome is not symmetric. However, the converse does not hold. An outcome outside the core may be a singleton farsighted stable set.

We begin with the definitions of the core in our model and its relative interior.

Definition 3
- *We say that an outcome $(S, x) \in X$ is in the core if there exists no $Q \in \mathcal{N}$ and $(T, y) \in X$ such that $(S, x) \to_Q (T, y)$ and $y_i > x_i$ for all $i \in Q$.*
- *We say that an outcome $(S, x) \in X$ is in the relative interior of the core if there exists no $Q \in \mathcal{N}$ and $(T, y) \in X$ such that $(T, y) \neq (S, x)$, $(S, x) \to_Q (T, y)$, and $y_i \geq x_i$ for all $i \in Q$.*

We denote C the core and $\mathring{C}$ the relative interior of the core.

The following proposition characterizes a condition for the nonemptiness of $\mathring{C}$. Similar results are shown by Watanabe and Muto (2008) and Hirai and Watanbe (2018), though the models are slightly different from the present paper, and they showed a condition for the nonemptiness of cores.

Proposition 1 *The relative interior of the core is nonempty if and only if* $\{n\} = B(\varepsilon)$ *for some* $\varepsilon \in E$*. Moreover,* $(S, x) \in \mathring{C}$ *implies that* $S = N$*,* $x_0 > 0$*, and* $x_i > L(0)$ *for all* $i \in S$*.*

Proof. We begin with the latter part. It is easy to see that $(\emptyset, x^{\emptyset}) \notin \mathring{C}$ because $(\emptyset, x^{\emptyset}) \to_{\{0\}\cup N} (N, x)$, where $x_0 = n(W(n) - L(0)) > 0 = x_0^{\emptyset}$ and $x_i = L(0)$ for all $i \in N$ by Assumption 1(i). Fix an arbitrary $(\tilde{S}, \tilde{x}) \in \mathring{C}$, where $\tilde{S} \neq \emptyset$. If $\tilde{x}_0 \leq 0$, then $(\tilde{S}, \tilde{x}) \to_{\{0\}} (\emptyset, x^{\emptyset})$ and $\tilde{x}_0 \leq 0 = x_0^{\emptyset}$, contradicting that $(\tilde{S}, \tilde{x}) \in \mathring{C}$. Thus, $\tilde{x}_0 > 0$. If $\tilde{x}_i \leq L(0)$ for some $i \in \tilde{S}$, then $(\tilde{S}, \tilde{x}) \to_{\{i\}} (\emptyset, x^{\emptyset})$ and $\tilde{x}_i \leq L(0) = x_i^{\emptyset}$, contradicting that $(\tilde{S}, \tilde{x}) \in \mathring{C}$. Thus, $\tilde{x}_i > L(0)$ for all $i \in \tilde{S}$. Next, suppose that $\tilde{S} \neq N$. By $\tilde{S} \neq \emptyset$, we can pick $j \in \tilde{S}$ and $j' \in N \setminus \tilde{S}$. Note that $\tilde{x}_{j'} = L(\tilde{s})$. Define $\tilde{T} = (\tilde{S} \setminus \{j\}) \cup \{j'\}$. Define $\tilde{y} \in \mathbb{R}^{n+1}$ such that $\tilde{y}_i = \tilde{x}_i$ for all $i \in \{0\} \cup (\tilde{S} \setminus \{j\})$. $\tilde{y}_i = L(\tilde{t})$ for all $i \in N \setminus \tilde{T}$, and $\tilde{y}_{j'} = \tilde{x}_j > L(0)$. Then, $(\tilde{S}, \tilde{x}) \to_{\{0\}\cup\tilde{T}} (\tilde{T}, \tilde{y})$, $\tilde{y}_i = \tilde{x}_i$ for all $i \in \{0\} \cup (\tilde{T} \setminus \{j'\})$, and $\tilde{y}_{j'} > L(0) > L(\tilde{s}) = \tilde{x}_{j'}$. This contradicts that $(\tilde{S}, \tilde{x}) \in \mathring{C}$. Hence, $\tilde{S} = N$.

We turn to the former part. First, assume that there exists some $\varepsilon^* \in E$ such that $\{n\} = B(\varepsilon^*)$. Let $x^* \in \mathbb{R}^{n+1}$ be such that $x_0^* = n(W(n) - L(0) - \varepsilon^*)$ and $x_i^* = L(0) + \varepsilon^*$ for all $i \in N$. Note that $(N, x^*) \in X$. We show that $(N, x^*) \in \mathring{C}$. Suppose that $(N, x^*) \notin \mathring{C}$. Then, there exists some nonempty $Q \subseteq \{0\} \cup N$ and (R, z) such that $(N, x^*) \to_Q (R, z)$ and $z_i \geq x_i^*$ for all $i \in Q$. Then, $0 \in Q$ since $0 \notin Q$ implies that $(R, z) = (\emptyset, x^{\emptyset})$, and thus, $z_i < x_i^*$ for all $i \in Q$. Note that $R = Q \setminus \{0\}$. Then,

$$z_0 = rW(r) - \sum_{i \in R} z_i \leq r(W(r) - L(0) - \varepsilon^*) < n(W(n) - L(0) - \varepsilon^*) = x_0^*$$

by $\{n\} = B(\varepsilon^*)$. This contradicts $z_0 \geq x_0^*$. Hence, $(N, x^*) \in \mathring{C}$.

Next, assume that $\{n\} \neq B(\varepsilon)$ for any $\varepsilon \in E$. We show that $\mathring{C} = \emptyset$. Suppose that there exists some $(S, y) \in \mathring{C}$. Note that we have already shown that $S = N$, $y_0 > 0$, and $y_i > L(0)$ for all $i \in N$ in the first part of this proof. Define $\varepsilon' = \left(\sum_{i \in N} y_i - nL(0)\right)/n$. Then, $\varepsilon' \in E$ since

$$n(W(n) - L(0) - \varepsilon') = nW(n) - nL(0) - \left(\sum_{i \in N} y_i - nL(0)\right) = nW(n) - \sum_{i \in N} y_i = y_0 > 0.$$

By $\{n\} \neq B(\varepsilon')$, there exists some $t = 1, ..., n-1$ such that $t \in B(\varepsilon')$. Let $\rho : N \to N$ be a permutation of N such that $y_{\rho(1)} \leq y_{\rho(2)} \leq \cdots \leq y_{\rho(n)}$. Let $T = \{\rho(1), ..., \rho(t)\}$. Define $z \in \mathbb{R}^{n+1}$ such that $z_{\rho(i)} = y_{\rho(i)}$ for all $i = 1, ..., t$, $z_{\rho(j)} = L(t)$ for all $j = t+1, ..., n$, and $z_0 = tW(t) - \sum_{i=1}^{t} z_{\rho(i)}$. Then, $(T, z) \in X$ and $(N, y) \to_{\{0\}\cup T} (T, z)$. We claim that

$z_0 \geq y_0$. Note that

$$\sum_{i=1}^{t} z_{\rho(i)} = \sum_{i=1}^{t} y_{\rho(i)} \leq \frac{t \sum_{i \in N} y_i}{n} = t(L(0) + \varepsilon').$$

Then,

$$z_0 = tW(t) - \sum_{i=1}^{t} z_{\rho(i)} \geq t(W(t) - L(0) - \varepsilon') \geq n(W(n) - L(0) - \varepsilon') = y_0$$

by $t \in B(\varepsilon')$. This contradicts that $(N, y) \in \mathring{C}$ together with $z_{\rho(i)} = y_{\rho(i)}$ for all $i = 1, ..., t$. Hence, $\mathring{C} = \emptyset$. ■

Note that the necessary and sufficient condition for the nonemptiness of $\mathring{C}$ is same as that for the existence of a symmetric and singleton farsighted stable set.

Next, we show that the relative interior of the core is a subset of the union of singleton farsighted stable sets.

Proposition 2 *For any $(N, x) \in \mathring{C}$, $\{(N, x)\}$ is a farsighted stable set.*

Proof. Fix an arbitrary $(N, x) \in \mathring{C}$. Note that $x_0 > 0$ and $x_i > L(0)$ for all $i \in N$ by the latter part of Proposition 1. To prove that $\{(N, x)\}$ is a farsighted stable set, it suffices to show its external stability. It is straightforward that $(N, x) \succ (\emptyset, x^{\emptyset})$ by $(\emptyset, x^{\emptyset}) \to_{\{0\} \cup N} (N, x)$, $x_0 > 0 = x_0^{\emptyset}$, and $x_i > L(0) = x_i^{\emptyset}$ for all $i \in N$. Fix an arbitrary $(T, y) \in X \setminus \{(N, x), (\emptyset, x^{\emptyset})\}$. Note that $T \neq \emptyset$ by $(T, y) \neq (\emptyset, x^{\emptyset})$. Then, $(N, x) \to_Q (T, y)$ implies that $Q = \{0\} \cup T$. By $(N, x) \in \mathring{C}$, there exists some $j \in \{0\} \cup T$ such that $x_j > y_j$. Then, $(T, y) \to_{\{j\}} (\emptyset, x^{\emptyset}) \to_{\{0\} \cup N} (N, x)$ yield $(N, x) \succ (T, y)$ by $x_j > y_j$, $x_0 > 0 = x_0^{\emptyset}$, and $x_i > L(0) = x_i^{\emptyset}$ for all $i \in N$. Thus, $\{(N, x)\}$ is externally stable. Hence, $\{(N, x)\}$ is a farsighted stable set. ■

Note that Proposition 2 also shows that the existence of an asymmetric farsighted stable set because the relative interior of the core usually includes an asymmetric outcome, though Theorems 1 and 2 says nothing for asymmetric farsighted stable set.

We turn to showing that there may exist a farsighted stable set $\{(N, x)\}$ such that (N, x) is not even in the core via an example.

Example 1 Let $N = \{1, 2\}$. Assume that $2(W(2) - L(0)) > W(1) - L(0) > 0$. Let $\varepsilon > 0$ be a sufficiently small real number such that $2(W(2) - L(0)) > W(1) - L(0) + \varepsilon$ and $L(0) - \varepsilon > L(1)$.

Define $x^* = (2(W(2) - L(0)), L(0) + \varepsilon, L(0) - \varepsilon)$. Obviously, $(N, x^*) \in X$. We have that $(N, x^*) \notin C$ because $(N, x^*) \rightarrow_{\{2\}} (\emptyset, x^\emptyset)$ and $x_2^\emptyset = L(0) > x_2^*$.

We show that $\{(N, x^*)\}$ is a farsighted stable set. Internal stability is obvious since it is a singleton. Thus, we show its external stability. We first show that $(N, x^*) \succ (\emptyset, x^\emptyset)$. This indirect dominance relation is yielded by $(\emptyset, x^\emptyset) \rightarrow_{\{0,1\}} (\{1\}, (W(1) - L(0), L(0), L(1))) \rightarrow_{\{0,1,2\}} (N, x^*)$ because $x_0^* = 2(W(2)-L(0)) > W(1)-L(0) > 0 = x_0^\emptyset$, $x_1^* = L(0) + \varepsilon > L(0) = x_1^\emptyset$, and $x_2^* = L(0) - \varepsilon > L(1)$. Hence, $(N, x^*) \succ (\emptyset, x^\emptyset)$.

Fix an arbitrary $(S, x) \in X \setminus \{(\emptyset, x^\emptyset), (N, x^*)\}$. Note that $S \neq \emptyset$. Consider the case where $x_0 < 2(W(2) - L(0)) = x_0^*$. In this case, $(S, x) \rightarrow_{\{0\}} (\emptyset, x^\emptyset) \rightarrow_{\{0,1\}} (\{1\}, (W(1) - L(0), L(0), L(1))) \rightarrow_{\{0,1,2\}} (N, x^*)$ yield $(N, x^*) \succ (S, x)$ by $x_0^* > x_0$ and $(N, x^*) \succ (\emptyset, x^\emptyset)$.

Assume, therefore, that $x_0 \geq 2(W(2) - L(0)) = x_0^*$ hereafter. Consider the case where $S = \{i\}$ for some $i = 1, 2$. In this case,

$$x_i = W(1) - x_0 \leq W(1) - 2(W(2) - L(0)) < L(0) - \varepsilon \leq x_i^*.$$

Thus, $(S, x) \rightarrow_{\{i\}} (\emptyset, x^\emptyset) \rightarrow_{\{0,1\}} (\{1\}, (W(1) - L(0), L(0), L(1))) \rightarrow_{\{0,1,2\}} (N, x^*)$ yield $(N, x^*) \succ (S, x)$ by $x_i < x_i^*$ and $(N, x^*) \succ (\emptyset, x^\emptyset)$.

Assume additionally that $S = N$ hereafter. If $x_1 < L(0) + \varepsilon = x_1^*$, then $(N, x) \rightarrow_{\{1\}} (\emptyset, x^\emptyset) \rightarrow_{\{0,1\}} (\{1\}, (W(1) - L(0), L(0), L(1))) \rightarrow_{\{0,1,2\}} (N, x^*)$ yield $(N, x^*) \succ (N, x)$ by $x_1^* > x_1$ and $(N, x^*) \succ (\emptyset, x^\emptyset)$. Therefore, assume further that $x_1 \geq L(0) + \varepsilon = x_1^*$ hereafter. By $x \neq x^*$, either $x_0 > x_0^*$ or $x_1 > x_1^*$. Then, $x_2 < x_2^* = L(0) - \varepsilon$. Then, $(N, x) \rightarrow_{\{2\}} (\emptyset, x^\emptyset) \rightarrow_{\{0,1\}} (\{1\}, (W(1) - L(0), L(0), L(1))) \rightarrow_{\{0,1,2\}} (N, x^*)$ yield $(N, x^*) \succ (S, x)$ by $x_2^* > x_2$ and $(N, x^*) \succ (\emptyset, x^\emptyset)$. Hence, $\{(N, x^*)\}$ is externally stable.

Therefore, the union of singleton farsighted stable sets is possibly strictly larger than $\mathring{C}$. On the other hand, the equivalence between singleton farsighted stable sets and the relative interior of the core is recovered if we restrict our attention to symmetric farsighted stable sets.

Proposition 3 *Let $(S, x) \in \bar{X}$. Then, $\{(S, x)\}$ is a farsighted stable set if and only if $(S, x) \in \mathring{C}$.*

Proof. The sufficiency follows from Proposition 2. Thus, we show the necessity.

Fix an arbitrary $(S, x) \in \bar{X}$. Assume that $\{(S, x)\}$ is a farsighted stable set. Suppose that $(S, x) \notin \mathring{C}$. Then, there exist $(T, y) \in X \setminus \{(S, x)\}$ and $P \subseteq \{0\} \cup N$ such that $(S, x) \rightarrow_P (T, y)$ and $y_i \geq x_i$ for all $i \in P$. Since $\{(S, x)\}$ is a farsighted stable set, $(S, x) \succ (T, y)$. Thus, there exist sequences of outcomes $(Q^0, z^0), ..., (Q^m, z^m)$ and coalitions $R^1, ..., R^m$ such that $(Q^0, z^0) = (T, y)$, $(Q^m, z^m) = (S, x)$, and for all $h = 1, ..., m$, $(Q^{h-1}, z^{h-1}) \rightarrow_{R^h} (Q^h, z^h)$ and $z_i^{h-1} < x_i$ for all $i \in R^h$.

First, consider the case where $T \neq \emptyset$. Then, $P = \{0\} \cup T$ by Assumption 2(ii). By $y_i \geq x_i$ for all $i \in \{0\} \cup T$, $R^1 \cap (\{0\} \cup T) = \emptyset$. It follows that $R^1 = \emptyset$ by Assumption 2. This contradicts that R^1 is a coalition.

Next, consider the case where $T = \emptyset$. Note that $(T, y) = (\emptyset, x^{\emptyset})$. By $(S, x) \neq (T, y)$, $S \neq \emptyset$. Then, $P \subseteq \{0\} \cup S$ by Assumption 2(i). Assume that $0 \in P$. Then, $x_0^{\emptyset} = 0 = y_0 \geq x_0$. Thus, $(Q^1, z^1) = (\emptyset, x^{\emptyset})$. For each $h = 2, ..., m$, if $(Q^{h-1}, z^{h-1}) = (\emptyset, x^{\emptyset})$, then $0 \notin R^h$ and $(Q^h, z^h) = (\emptyset, x^{\emptyset})$ by $x_0^{\emptyset} \geq x_0$. Thus, $(Q^m, z^m) = (\emptyset, x^{\emptyset})$. This contradicts the nonemptiness of S. Assume therefore that $0 \notin P$. Thus, $P \subseteq S$. By the symmetry of (S, x) and $L(0) = y_i \geq x_i$ for all $i \in P$, we have that $x_i^{\emptyset} = L(0) \geq x_i$ for all $i \in S$. Moreover, $x_i^{\emptyset} = L(0) > L(s) \geq x_i$ for all $i \in N \setminus S$. Thus, $R^1 = \{0\}$. It follows that $(Q^1, z^1) = (\emptyset, x^{\emptyset})$. For each $h = 2, ..., m$, if $(Q^{h-1}, z^{h-1}) = (\emptyset, x^{\emptyset})$, then $R^h = \{0\}$ and $(Q^h, z^h) = (\emptyset, x^{\emptyset})$ by $x_i^{\emptyset} = L(0) \geq x_i$ for all $i \in N$. Thus, $(Q^m, z^m) = (\emptyset, x^{\emptyset})$. This contradicts the nonemptiness of S. Hence, $(S, x) \in \mathring{C}$. ■

5 Concluding remarks

This paper offered a complete characterization of symmetric farsighted stable sets of an abstract game representing patent licensing negotiations by farsighted players. An exogenously given net profit of licensee firms plays an important role in this characterization. A symmetric farsighted stable set is characterized by a set of outcomes where the patent holder maximizes own profit provided that each licensee firm is allowed to enjoy a given net profit. We conclude the paper with a remark.

The profit of a patent holder is one of the main concerns in patent licensing. From our characterization, it can be easily derived that the supremum of a patent holder's profit supported by symmetric farsighted stable set is $\max_{s=1,...,n} s(W(s) - L(0))$, while the infimum is 0. The supremum is consistent with the results for the bargaining set, kernel, and nucleolus (Watanabe and Muto, 2008; Kishimoto and Watanabe, 2017) at least for the case where $s(W(s) - L(0))$ is not maximized at $s = n$. On the other hand, the infimum shows that our formulation of negotiations by farsighted players may allow lower values than the results in negotiations employed in the literature. A more sophisticated comparison of the patent holder's profits with the literature may follow.

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