ソシオネットワーク戦略ディスカッションペーパーシリーズ ISSN 1884-9946
第 48 号　2017 年 7 月
RISS Discussion Paper Series
No.48　July, 2017

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Acyclic Rational Choice with Indifference-Transitivity*

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July 2017

Abstract

This study investigates an acyclic and indifference-transitive (AC-IT) rational choice function, a choice function that is rationalizable with a complete, acyclic and indifference transitive relation.

New axioms, *recursivity under union* and its mates, together with other well known axioms characterize AC-IT rational choice functions defined on the full, base, and arbitrary domains.

The relationship between AC-IT rational choice functions and two equilibrium concepts in cooperative games (strict core and von Neuman-Morgenstern's stable set) is also investigated.

Keywords: acyclicity, indifferece-transitivity, choice functions, rationalization

*We thank two anonymous referees of the journal. This paper was presented at the 12th meeting of the Society for Social Choice and Welfare, Boston, June 2014. We thank all the participants of the meeting. This work was financially supported by JSPS KAKENHI, Grant-in-Aid for Challenging Exploratory Research, Grant Number 25590049. The authors would like to thank Enago (www.engo.jp) and Konan Translation Service (www.konan-tr-s.com/company) for the English language review.

1 Introduction

An acyclic and indifference-transitive (AC-IT) relation —an acyclic binary relation with its indifference part being transitive— has become of more importance in the Arrovian social choice literature since Iritani, Kamo and Nagahisa [11] proved an impossibility theorem. It says that there exists a unique vetoer for any AC-IT valued social choice rule satisfying Pareto and Arrow's independence of irrelevant alternatives when there are at least four alternatives. This result stimulates the study of AC-IT rational choice functions, which is the subject of the present paper.

In addition to this, there exists another motivation to study AC-IT rational choice. We show that an AC-IT preference can be derived from two complete and transitive preferences; it inherits its symmetric parts from one and its asymmetric parts from the other (Proposition 1). AC-IT preferences are therefore concluded to be 'rational' in the sense that it is made by crossing two transitive preferences in a consistent way.

A choice function is *AC-IT rational* if and only if it is rationalizable with a complete and AC-IT relation. Three types of domain of choice functions are employed: full, base, and arbitrary domains. Let a universal finite set of alternatives be given. The full domain is the domain of a choice function consisting of *all* nonempty subsets of the universal set. A base domain is a domain containing all one-element and all two-element sets of the universal set. An arbitrary domain is a domain consisting of *some*, *but not necessarily all*, nonempty subsets of the universal set.

In the literature, full domain case has been studied; by Arrow [2], Uzawa [28], Sen [20], [21], Plott [13], Jamison and Lau [12], Schwartz [19] and Blair et al. [4]. The study of arbitrary domain case dates back to the field of consumer theory with revealed preference (e.g., Samuelson [17], [18], Houthakker [10]). The subject of choice functions with base and arbitrary domains has been studied by Richter [15], [16], Hansson [8], Suzumura [22], [24], [25] and Bossert et al. [5]. Despite its long history, the study of choice functions has not dealt with AC-IT case for any of the three domains.

The first purpose of this study is to characterize AC-IT rational choice functions with several choice-consistency axioms. We establish the following results.

(1) Full domain: Blair et al. [4] showed that a choice function is acyclic rational, that is, rationalizable with a complete and acyclic relation, if and only if it satis-

fies Chernoff's axiom (**CA**) and Generalized Condorcet property (**GC**). Since AC-IT rationality trivially implies acyclic rationality, their result suggests that one more axiom is necessary for the characterization of AC-IT rational choice functions. We demonstrate that *Recursivity under union* (**RU**) is just the one (Theorem 1). Theorem 1 says that a choice function is AC-IT rational if and only if it satisfies **CA**, **GC**, and **RU**.

(2) Base domain: Bossert et al. [5] showed that a choice function is acyclic rational if and only if it satisfies Richter [16]'s D-congruence (**DC**) and an acyclicity condition of a revealed preference (**NEcC**). This result suggests that one more axiom is necessary for AC-IT rational. We show that a choice function is AC-IT rational if and only if it satisfies the two axioms and *conditional **RU*** (**CRU**), a natural extension of **RU** (Theorem 3).

(3) Arbitrary domain: Bossert et al. [5] showed that if a choice function satisfies Strong A-congruence (**SAC**), then it is acyclic rational. Their result suggests that one more axiom is necessary for AC-IT rational. We show that if a choice function satisfies this axiom and *strong conditional **RU*** (**SCRU**), a strong version of **CRU**, then it is AC-IT rational (Theorem 4).

An intuitive meaning of **RU** is as follows: There are two competitions, competition 1 and competition 2. **RU** says that if there is a winner in both of the competitions, then there is no need for any further competition among the winners. If it were done, the result would be recursive; no winner would miss in the further competition. This is because every winner of competition 1 is equally matched with the competitor, who won both the competitions, and he/she is equally matched with every winner of competition 2. Since "equally matched" relation is thought of as a transitive indifference relation, no one can tell which of the two winners is stronger. This illustration suggests the existence of a close association of **RU** with indifference-transitivity.

A choice function is full rational if and only if it is rationalizable with a complete and transitive relation. Axiomatizations of full rational choice functions were established by Arrow [2], Sen [20], [21], and Jamison and Lau [12]. Using **RU**, an alternative axiomatization of full rational choice functions can also be established; a choice function is full rational if and only if it satisfies **CA**, **RU** and Superset axiom

(**SUA**)[1](Theorem 2).

The second purpose of this study is to investigate the relationship between AC-IT rational choice functions and equilibrium concepts in an abstract game. We assume full domain here. It is well known in the literature that if the choice relation reads "dominate" relation in an abstract cooperative game, then there exists a close logical relationship between rational choice functions and equilibrium concepts of cooperative game, such as core or stable set. Wilson [30] established that the set of alternatives selected by an acyclic rational choice function always coincides with the core in the game. Plott [14] and Bandyopadhyay and Sengupta [3] showed that a choice function is quasi-transitive rational if and only if the core coincides with von Neumann-Morgenstern's stable set. In line with the above-mentioned research, this study shows that a choice function is AC-IT rational if and only if the set of alternatives selected by the choice function always coincides with the strict-core, which is a strong version of the core, and that the indifference closure of the strict core, defined by the set of alternatives each of which is indifferent to an alternative in the strict core, has the same property as that of von Neumann-Morgenstern's stable set (Theorem 5).

Finally, by converting Iritani Kamo and Nagahisa [11] into a social choice correspondence version, the counterparts of their vetoer theorem are also established (Theorems 6 and 7).

This study is organized as follows. Section 2 introduces notation and definitions. Section 3 investigates an interesting property of AC-IT preferences. Sections 4 and 5 state axiomatizations with the three domains. Section 6 explores the relationship between AC-IT rational choice functions and equilibrium concepts in an abstract game. Section 7 presents choice function counterparts to the impossibility theorems of AC-IT valued Arrovian social choice rules. Section 8 concludes. Subordinate matters of little importance relative to the main results are relegated to Appendix.

2 Notation and Definitions

Let X be the set of alternatives. Let Ω_F and Ω_B be the set of all nonempty subsets of X and the set of all one-element and all two-element sets of X respectively. A

[1]**SUA** was introduced by Blair et at. [4].

domain Ω is a nonempty subset of Ω_F. We say that Ω is the *full domain* if $\Omega = \Omega_F$, a *base domain* if $\Omega_B \subset \Omega$, and an *arbitrary domain* if there is no specific structure on Ω. A choice function C is a mapping defined on Ω such that $\emptyset \neq C(S) \subset S$ for all $S \in \Omega$.

A binary relation on X is denoted by $\succcurlyeq$ where $\succ$ and $\sim$ are asymmetric and symmetric part of $\succcurlyeq$ respectively. A binary relation $\succcurlyeq$ is complete if for any $x, y \in X$, either $x \succcurlyeq y$ or $y \succcurlyeq x$ holds. A binary relation $\succcurlyeq$ is

(i) *transitive* if for any $x, y, z \in X$, $x \succcurlyeq y$ and $y \succcurlyeq z$ imply $x \succcurlyeq z$;

(ii) *quasi-transitive* if for any $x, y, z \in X$, $x \succ y$ and $y \succ z$ imply $x \succ z$;

(iii) *indifference-transitive* if for any $x, y, z \in X$, $x \sim y$ and $y \sim z$ imply $x \sim z$; and

(iv) *acyclic* if for any $x_1, x_2, ..., x_{t-1}, x_t \in X$, $x_1 \succ x_2 \succ ... \succ x_{t-1} \succ x_t$ does not imply $x_t \succ x_1$.

Note that $\succcurlyeq$ is transitive if and only if it is quasi- and indifference-transitive, and that quasi-transitivity implies acyclicity. An acyclic and indifference-transitive relation is abreviated as an AC-IT relation.

Given a binary relation $\succcurlyeq$ on X and $S \in \Omega_F$, the set of $\succcurlyeq$-greatest elements of S, denoted $G(S, \succcurlyeq)$, is defined by $G(S, \succcurlyeq) = \{x \in S : x \succcurlyeq y \ \forall y \in S\}$. It is well known that $G(S, \succcurlyeq)$ is nonempty for all $S \in \Omega_F$ if and only if $\succcurlyeq$ is complete and acyclic on X.

Definition 1 A choice function C is *rationalizable* with $\succcurlyeq$ if $C(S) = G(S, \succcurlyeq)$ for all $S \in \Omega$. In this case we call $\succcurlyeq$ a *rationalization* of C.

A choice function is *full rational* (*FR*) if and only if it is rationalizable with a complete and transitive relation. A choice function is *AC-IT rational* (*AC-ITR*) if and only if it is rationalizable with a complete and AC-IT relation. *Quasi-transitive rationality* (*QTR*) and *acyclic rationality* (*ACR*) are defined as well.

Definition 2 Let C be a choice function. A *base relation* $\succcurlyeq_C$ for C is defined by[2]

$$x \succcurlyeq_C y \Longleftrightarrow x \in C(\{x, y\}).$$

[2] See Arrow [2], Herzberger [9].

Let $\succ_C$ and $\sim_C$ be asymmetric and symmetric part of $\succcurlyeq_C$ respectively. A choice function C defined on a base domain is rationalizable if and only if $\succcurlyeq_C$ is a rationalization of C[3]. A choice function C defined on the full domain is rationalizable if and only if it is acyclic rational.

3 AC-IT preferences

Prior to an axiomatic analysis of AC-IT rational choice functions, we show the motivation of the study. The discussion consists of two cases, a social preference case in which we require that social preference be AC-IT, and an individual preference case in which we require that individual preference be AC-IT.

The social preference case: As stated in the previous section, acyclicity together with completeness is a necessary and sufficient condition for the existence of a best element in every finite set. Regarding IT relations, some early psychological experiments demonstrated intransitivity of IT relations (see, for example, Tversky [27]). The problem here is the existence of psychological threshold, which prevents individuals from discerning one alternative from another. This criticism may be partly true for individual preferences. But it is also undoubtedly true that most of choice problems can escape from this difficulty by assuming that each alternative is fairly divided to be discriminated each other[4]. The requirement that social preferences be AC-IT is therefore quite natural if we wish to select best alternatives in the set of well discriminated alternatives.

The individual preference case: This case relates to rationality of individual preferences. A meaningful economic example of AC-IT preferences gives us some intuitive sense of how they differ from conventional preferences. Suppose that four candidates a, b, c and d intend to run for a city council position. You are considering whom you should vote for in the election. The information available to you is as follows:

- a, c, and d are members of Party X while b is a member of Party Y;
- You favor Party X in politics more than Party Y and you know a quite well as the best politician representing the political briefs of Party X;

[3]See Suzumura 1983 Lemma 2.2 p28. The assumption $\Omega = \Omega_F$ in this lemma can be relaxed to $\Omega_B \subset \Omega$.

[4]See Feldman and Seranno [7] p12.

- b has been a good friend of yours, and you have been personally attached to him;
- You prefer a to b since apart from your eager support for Party X, nothing strikes you more than a's admirable dedication to politics, which excels your long-lasting friendship with b; and
- c and d are total strangers.

It is possible that your choice is expressed by the following choice function:

(i) $C(\{a,b\}) = \{a\}$; You vote for a against b (if only a and b run for the position[5]).

(ii) $C(\{b,c\}) = C(\{b,d\}) = C(\{b,c,d\}) = \{b\}$; You vote b against c or d since you know b very well, but not c and d.

(iii) $C(\{a,c\}) = \{a,c\}$, $C(\{a,d\}) = \{a,d\}$, $C(\{c,d\}) = \{c,d\}$, and $C(\{a,c,d\}) = \{a,c,d\}$; You are indifferent concerning the differences between the candidates of Party X.

(iv) $C(\{a,b,c\}) = C(\{a,b,d\}) = C(\{a,b,c,d\}) = \{a\}$; This follows from (i)-(iii).

This choice function is rationalizable with the following AC-IT but not quasi-transitive preference:

$$a \succ b \succ c,\ \ b \succ d,\ \ a \sim c \sim d \sim a$$

This example illustrates that an AC-IT preference is a composition of two transitive preferences, one of which is determined only by party affiliation, and the other determined only by natural feeling such as respect for others or personal closeness. That is,

$$\begin{array}{lccl} \succcurlyeq_1 & \text{(party affiliation preference)} & : & a \sim_1 c \sim_1 d \succ_1 b \\ \succcurlyeq_2 & \text{(natural feeling preference)} & : & a \succ_2 b \succ_2 c \sim_2 d \end{array}$$

Note that which of the two preferences becomes dominant in the choice problems depends on whether b is available or not. If b is not available, you make a choice without being influenced by your emotions. But if b becomes available, you must feel the emotion such as attachment to friends or respect to others rise in your mind

[5]This is the interpretation of $\{a,b\}$ in $C(\{a,b\})$.

and try to evaluate more seriously the candidates. There seem to be many choice problems in which the choice changes when a totally different alternative becomes available. We show more examples such as a competition of a couple of western realistic landscape paintings and an ukiyoe landscape painting, and a marriage problem in which candidates are a few middle class Japanese young men and an old Arabian millionaire[6].

Note also that the indifference sets of the AC-IT preference are determined by party affiliation while the preference between candidates belonging to parties different with each other is determined by natural feelings. This observation suggests the following proposition.

Let $\succcurlyeq_1$ and $\succcurlyeq_2$ be binary relations on X. A composition of $\succcurlyeq_1$ and $\succcurlyeq_2$is the binary relation $\succcurlyeq_{com}$ on X such that for any $x, y \in X$,

(i) $x \sim_{com} y$ if and only if $x \sim_1 y$;

(ii) $x \succ_{com} y$ if and only if $[x \succ_1 y$ and $x \sim_2 y]$ or $[x \nsim_1 y$ and $x \succ_2 y]$.

In the example, it is easy to check that the AC-IT preference is a composition of the party affiliation preference $\succcurlyeq_1$ and the natural feeling preference $\succcurlyeq_2$.

Based on our composition rule, we give a characterization of AC-IT preference relations. Proposition 1 says that the composition of complete and transitive preferences must be AC-IT. Proposition 2 says that any AC-IT preference can be regarded as the composition of some complete and transitive preferences.

Proposition 1 *If $\succcurlyeq_1$ and $\succcurlyeq_2$ are complete and transitive on X, then their composition $\succcurlyeq_{com}$ is necessarily complete and AC-IT on X.*

Proof. It is obvious that $\succcurlyeq_{com}$ is complete and indifference-transitive. We show that $\succcurlyeq_{com}$ is acyclic. Suppose that a finite sequence $x_1, ..., x_n \in X$ satisfies

$$x_k \succ_{com} x_{k+1} \text{ for } k = 1, ..., n-1.$$

Then for each $k = 1, ..., n-1$, either of the following properties holds:

(a) $x_k \succ_1 x_{k+1}$ and $x_k \sim_2 x_{k+1}$.

(b) $x_k \nsim_1 x_{k+1}$ and $x_k \succ_2 x_{k+1}$.

[6]It may be helpful to refer to Ariely ([1], Chapter 1) where many actual examples and experiments illustrate that people's preferences between two alternatives may change when a third alternative, called decoy, becomes available.

If (a) holds for all k, then transitivity of $\succsim_1$ and $\succsim_2$ implies that $x_1 \succ_1 x_n$ and $x_1 \sim_2 x_n$, which means that $x_1 \succ x_n$. If there exists some k such that (b) holds, then transitivity of $\succsim_2$ implies $x_1 \succ_2 x_n$. It follows from the definition of $\succsim_{com}$that $x_1 \sim_{com} x_n$ if $x_1 \sim_1 x_n$, and $x_1 \succ_{com} x_n$ otherwise. Therefore we obtain that $x_1 \succsim_{com} x_n$. ■

Proposition 2 *For any complete and AC-IT relation $\succsim$ on X, there exist complete and transitive relations, $\succsim_1$ and $\succsim_2$, such that $\succsim$ is the composition of $\succsim_1$ and $\succsim_2$.*

Proof. If $\succsim$ is trivial, i.e., $x \sim y$ for all $x, y \in X$, then the result is obvious; We set $\succsim_1 = \succsim_2 = \succsim$. In the following, we assume that $\succsim$ is nontrivial, i.e., $x \nsim y$ for some $x, y \in X$.

Let $\mathcal{T} = \{I_\lambda\}_{\lambda \in \Lambda}$ be the indifference classes with respect to the relation $\sim$. Note that $\mathcal{T}$ is a partition of X, i.e., $X = \bigcup_{\lambda \in \Lambda} I_\lambda$ and $I_\lambda \cap I_\mu \neq \emptyset$ implies $I_\lambda = I_\mu$. Define the binary relation $\trianglerighteq$ on $\mathcal{T}$ as follows: For any $I_\lambda, I_\mu \in \mathcal{T}$, $I_\lambda \trianglerighteq I_\mu$ if and only if there exists $x \in I_\lambda$ such that $x \succsim y$ for all $y \in I_\mu$. It is easy to see that $\trianglerighteq$ is reflexive [7] and transitive. It follows from Szpilrajn extension theorem ([26]) that $\trianglerighteq$ has a complete extension $\trianglerighteq^*$. Then we define the binary relation $\succsim_1$ on X as follows: For any $x, y \in X$, $x \succsim_1 y$ if and only if $I_\lambda \trianglerighteq^* I_\mu$ whenever $x \in I_\lambda$ and $y \in I_\mu$. It is straightforward that $\succsim_1$ is complete and transitive.

Consider the asymmetric part $\succ$ of $\succsim$. Since $\succsim$ is acyclic, it is easy to see that $\succ$ is *consistent*, that is, for any $x, y \in X$, if there exists a sequence $x_1, ..., x_n$ such that

$$x = x_1 \succ \cdots \succ x_n = y,$$

then $y \succ x$ does not hold. Applying Theorem 3 of Suzumura ([23]), we obtain the complete and transitive extension of $\succ$, which is denoted by $\succsim_2$. Note that for any $x, y \in X$, $x \succ y$ implies $x \succ_2 y$.

Let $\succsim_{com}$ be the composition of $\succsim_1$ and $\succsim_2$. We show that $\succsim = \succsim_{com}$. Since the definition of $\succsim_{com}$ directly implies that $\sim = \sim_{com}$, it suffices to show that $\succ = \succ_{com}$. Take x, $y \in X$ arbitrarily. If $x \succ y$, then $x \nsim_1 y$ and $x \succ_2 y$, which implies that $x \succ_{com} y$. Conversely, if $x \succ_{com} y$, then it follows from the definition of $\succ_{com}$ that $x \succsim_2 y$. Since $\succsim_2$is the extension of $\succ$, $y \succ x$ does not hold, which means $x \succsim y$. Since

[7] A relation $\succsim$ on X is reflective if and only if $x \succsim x$ for all $x \in X$.

$x \not\sim y$ by the supposition,[8] we obtain $x \succ y$. ■

Proposition 2 looks more appealing if each of the two transitive preferences is defined in a consistent way. Take a look at the city council example again. The symmetric parts of party affiliation preference have a simple reason: x is indifferent to y if and only if they belong to the same party. The asymmetric parts of natural feeling preference do so as well: x is preferred to y if and only if you appreciate x as a man of dignity in many respects more than you do for y. Proposition 2 thus says that the AC-IT preference inherits two good characteristics from the two transitive preferences, inheriting its symmetric parts from the first and its asymmetric parts from the second.

4 Axiomatizations with full domain

We assume $\Omega = \Omega_F$ throughout this section. First we list four axioms.

Recursivity under union (RU) For all $S_1, S_2 \in \Omega_F$, if $C(S_1) \cap C(S_2) \neq \emptyset$, then $C(S_1) \cup C(S_2) = C(C(S_1) \cup C(S_2))$.

Chernoff's axiom (CA) (Sen [20]) For all $S_1, S_2 \in \Omega_F$, if $S_1 \subset S_2$ and $S_1 \cap C(S_2) \neq \emptyset$, then $S_1 \cap C(S_2) \subset C(S_1)$.

Generalized Condorcet's axiom (GC) (Plott [13]) For all $S \in \Omega_F : G(S, \succcurlyeq_C) \subset C(S)$.

Superset axiom (SUA) (Blair et al. [4]) For all $S_1, S_2 \in \Omega_F$, if $S_1 \subset S_2$ and $C(S_1) \supset C(S_2)$, then $C(S_1) = C(S_2)$.

RU is the new axiom we present, which is the major topic of discussion in this section. It says that if the two choice sets, $C(S_1)$ and $C(S_2)$, have a nonempty intersection, the choice from the union of the two sets is recursive. **RU** is interpreted as follows: Suppose that there are two competitions, competition 1 and competition 2. Let $C(S_1)$ be the set of the winners in competition 1, and $C(S_2)$ be the set of the winners in competition 2. If there is a competitor who participated in and won both the competitions, it is of no use holding any further competition among the

[8] If $x \sim y$, then $x \sim_{com} y$, a contradiction.

winners of the two competitions. If this were to be carried out, then the result would be recursive; no winner would miss in the further competition. Every winner in competition 1 is equally matched with the competitor, who is also equally matched with every winner in competition 2. It is therefore difficult to say which of the two winners is better. This illustration suggests a close association of **RU** with indifference-transitivity. Lemma 1 below shows that this suggestion is true if **RU** is combined with **CA**.

Lemma 1 *If a choice function C satisfies **CA** and **RU**, then $\sim_C$ is transitive.*

Proof. Let $x, y, z \in X$ be such that $x \sim_C y \sim_C z$. By definition of $\sim_C$, we have $C(\{x, y\}) = \{x, y\}$ and $C(\{y, z\}) = \{y, z\}$. Applying **RU**, we have $C(\{x, y, z\}) = \{x, y, z\}$. Since $\{x, z\} = \{x, z\} \cap C(\{x, y, z\}) \neq \emptyset$, **CA** is applied, then we have $\{x, z\} = C(\{x, z\})$, which means $x \sim_C z$, the desired result. ∎

In contrast, Lemma 2 below shows that **CA** alone is responsible for acyclicity of $\succcurlyeq_C$ and that **SUA** together with **CA** is responsible for transitivity of $\succ_C$.

Lemma 2 *Let a choice function C satisfying **CA** be given. Then the followings are true:*

(i) $C(S) \subset G(S, \succcurlyeq_C)$ for all $S \in \Omega_F$;

(ii) $\succcurlyeq_C$ is acyclic; and

*(iii) if C satisfies **SUA**, then $\succ_C$ is transitive.*

Proof. (i) Take $x \in C(S)$ and $y \in S$ arbitrarily. Since $x \in \{x, y\} \cap C(S) \neq \emptyset$, **CA** is applied so that $x \in C(\{x, y\})$, which means $x \succcurlyeq_C y$ for all $y \in S$, the desired result.

(ii) On the contrary, suppose that there is a cycle such that $x_1 \succ x_2 \succ ... \succ x_t \succ x_1$. This together with (i) means $C(\{x_1, x_2, ..., x_t\}) \subset G(\{x_1, x_2, ..., x_t\}, \succcurlyeq_C) = \emptyset$, a contradiction.

(iii) Let $x, y, z \in X$ be such that $x \succ_C y \succ_C z$. (i) and (ii) imply that $\{x\} = C(\{x, y, z\})$ and $x \in C(\{x, z\})$. Thus, **SUA** is applied, and we have $\{x\} = C(\{x, z\})$, that is, $x \succ_C z$, which is the desired result. ∎

We show that **RU** with the help of the other three axioms can give an axiomatization of AC-IT rational choice functions and an alternative axiomatization of full rational choice functions[9].

Theorem 1 *A choice function C is AC-IT rational if and only if it satisfies **CA**, **RU** and **GC**.*

Proof. 'only if': Since AC-IT rational is AC rational, **CA** and **GC** were already established (Suzumura [25] Theorem 2.8 p35). Thus it suffices to show **RU**. Take $z \in C(S_1) \cap C(S_2)$, $x \in C(S_1)$ and $y \in C(S_2)$. (i) of Lemma 2 shows $x \sim_C z \sim_C y$ and therefore Lemma 1 shows $x \sim_C y$. Since x and y are arbitrarily taken, we have $G(C(S_1) \cup C(S_2), \succcurlyeq_C) = C(S_1) \cup C(S_2)$. Noting that $\succcurlyeq_C$ is a rationalization of C[10], we have $C(C(S_1) \cup C(S_2)) = G(C(S_1) \cup C(S_2), \succcurlyeq_C)$. Combining the two, we see **RU** hold.

'if' : Lemmas 1 and 2 show that $C(S) \subset G(S, \succcurlyeq_C)$ for all $S \in \Omega_F$, where $\succcurlyeq_C$ is an AC-IT relation. The inclusion relation $\supset$ follows from **GC**. ■

Theorem 2 *A choice function C is full rational if and only if it satisfies **CA**, **SUA** and **RU**.*

Proof. 'if': Let C be a choice function satisfying **CA**, **SUA** and **RU**. Since Lemmas 1 and 2 show that the base relation $\succcurlyeq_C$ is complete and transitive, the only thing to prove is that $\succcurlyeq_C$ is a rationalization of C, that is, $C(S) = G(S, \succcurlyeq_C)\ \forall S \in \Omega_F$. Take $x \in C(S)$ and $y \in S$ arbitrarily. Since $x \in \{x, y\} \cap C(S)$, **CA** is applied and hence $x \in C(\{x, y\})$, which completes

$$C(S) \subset G(S, \succcurlyeq_C). \tag{1}$$

Next we show

$$G(S, \succcurlyeq_C) = C(G(S, \succcurlyeq_C)) \tag{2}$$

If $G(S, \succcurlyeq_C)$ is singleton, (2) is obvious. Let $G(S, \succcurlyeq_C) = \{x_1, ..., x_k\}$ $(k \geq 2)$. By definition of $G(S, \succcurlyeq_C)$, we have $\{x_1, x_2\} = C(\{x_1, x_2\})$ and $\{x_2, x_3\} = C(\{x_2, x_3\})$.

[9] Other axiomatizations for full rational choice functions were established by Arrow [2], Sen ([20], [21]) and Jamison and Lau [12].

[10] See Definition 2.

RU is applied, and we have $\{x_1, x_2, x_3\} = C(\{x_1, x_2, x_3\})$. **RU** is applied again between $\{x_1, x_2, x_3\} = C(\{x_1, x_2, x_3\})$ and $\{x_3, x_4\} = C(\{x_3, x_4\})$, and we have $\{x_1, x_2, x_3, x_4\} = C(\{x_1, x_2, x_3, x_4\})$. This procedure continues until we get $\{x_1, .., x_k\} = C(\{x_1, .., x_k\})$, which completes the proof of (2).

Combining (1) and (2), we have $C(G(S, \succcurlyeq_C)) \supset C(S)$. Since $G(S, \succcurlyeq_C) \subset S$, **SUA** is applied and hence $C(G(S, \succcurlyeq_C)) = C(S)$. (2) is applied again, we have $G(S, \succcurlyeq_C) = C(S)$, which is the desired result.

only if : Since *FR* is *AC-ITR*, Theorem 1 shows that C satisfies **CA** and **RU**. It was already known that **SUA** is necessary for *QTR* (See Suzumura Th.2.6 p33), and hence **SUA** is necessary for *FR* as well. ■

The table below reviews the axiomatizations of *FR*, *QTR*, *AC-ITR*, and *ACR* choice functions. Note that any three of the four axioms can axiomatize *FR*, *QTR* and *AC-ITR* respectively.

	FR	*QTR*	*AC-ITR*	*ACR*
CA	■	■	■	■
SUA	■	■		
RU	■		■	
GC		■	■	■
	Th.2	Blair et al. [4]	Th.1	Blair et al. [4]

RU can be interpreted as a tournament condition as well as Path-independence (PI)[11], which says that for all $S_1, S_2 \in \Omega_F$, $C(S_1 \cup S_2) = C(C(S_1) \cup C(S_2))$. **RU** and PI however work differently. If **CA** is assumed, PI is known to be equivalent to **SUA**[12]. Lemma 1 shows that **SUA** is responsible for transitivity of $\succ_C$ whereas **RU** is responsible for transitivity of $\sim_C$.

GC in Theorem 1 can be replaced with the axiom below, which looks more appealing than **GC**:

Complete symmetry (CS) For all $S \in \Omega_F$ and all $x, y \in S$, the following two statements are equivalent:

(i) $x \in C(S)$ if and only if $y \in C(S)$.

[11]See Plott [13].

[12]See Blair et al.[4].

(ii) For all $z \in S$, $x \in C(\{x, z\})$ if and only if $y \in C(\{y, z\})$.

CS says that, for any x, y in S, they should be treated equally in choice if and only if they cannot be distinguished through the pair wise comparison to any alternative z in S.

Lemma 3 *For any choice function C satisfying **CA** and **RU**, C satisfies **GC** if and only if it satisfies **CS**.*

Proof. Suppose C satisfies **GC**. Theorem 1 says

$$C(S) = G(S, \succcurlyeq_C) \text{ where } \succcurlyeq_C \text{ is an AC-IT relation} \tag{3}$$

Then **CS** follows from (3) directly.

Suppose C satisfies CS. Lemmas 1 and 2 say

$$C(S) \subset G(S, \succcurlyeq_C) \text{ where } \succcurlyeq_C \text{ is an AC-IT relation.} \tag{4}$$

A direct application of **CS** to (4) shows that **GC** is satisfied. ■

The examples below illustrate the tightness of the axiomatizations. Examples 1, 2 and 3 serve Theorem 1 and Examples 1, 2 and 4 serve Theorem 2. We assume $X = \{x, y, z\}$ in all the examples.

Example 1 (CA) Let a choice function C be defined by $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, $C(\{z, x\}) = \{z\}$, and $C(\{x, y, z\}) = \{x, y, z\}$. It is obvious that C satisfies all the axioms except for **CA** and is not rationalizable.

Example 2 (RU) Let a choice function C be defined by $C(\{x, y\}) = \{x, y\}$, $C(\{y, z\}) = \{y, z\}$, $C(\{x, z\}) = \{x\}$, and $C(\{x, y, z\}) = \{x, y\}$. It is obvious that C satisfies all the axioms except for **RU** and is not AC-IT rational.

Example 3 (GC) Let a choice function C be defined by $C(\{x, y\}) = \{x, y\}$, $C(\{y, z\}) = \{y\}$, $C(\{x, z\}) = \{x\}$, and $C(\{x, y, z\}) = \{x\}$. It is obvious that C satisfies all the axioms except for **GC** and **SUA**[13] and is not rationalizable.

[13]It is impossible to make an example that satisfies all the axioms except for **GC**. See Theorem 2; by assuming **CA** and **RU**, **SUA** implies *FR*, which further implies **GC**.

Example 4 (SUA) Let a choice function C be defined by $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, $C(\{x, z\}) = \{x, z\}$, and $C(\{x, y, z\}) = \{x\}$. It is obvious that C satisfies all the axioms except for **SUA** and is not full rational.

See Appendix for further remarks on **RU**; a basic property of **RU** and the logical relationship of **RU** with other well-known choice consistency axioms.

5 Axiomatizations with base and arbitrary domains

Let an arbitrary domain Ω be given. Let us define the binary relations R_C, E_C and I_C on X as follows:

- $xR_Cy \Longleftrightarrow$There exists $S \in \Omega$ such that $x \in C(S)$ and $y \in S$.
- $xE_Cy \Longleftrightarrow$There exists $S \in \Omega$ such that $x \in C(S)$ and $y \in S\backslash C(S)$;
- $xI_Cy \Longleftrightarrow$There exists $S \in \Omega$ such that $x, y \in C(S)$.

The definitions of R_C and E_C are the same as in Bossert et al. [5]. I_C is a new definition.

We define a natural extension of **RU** when the domain is general.

Conditional RU (CRU) . For all $S_1, S_2 \in \Omega$, if $C(S_1) \cap C(S_2) \neq \emptyset$, then $C(T) = T$ for all $T \in \Omega$ satisfying $T \subset C(S_1) \cup C(S_2)$.

We will show later that **CRU** together with other two well known axioms can characterize AC-IT rational choice functions with base domains. However this result is not true for arbitrary domains. A stronger version of **CRU** is necessary for the purpose:

Strong conditional RU (SCRU) For any $x, y \in X$, if there is a finite sequence of alternatives $z_1, ..., z_t$ such that $xI_Cz_1I_Cz_2I_C \cdots I_Cz_tI_Cy$, then $x \in C(S) \Longleftrightarrow y \in C(S)$ for all $S \in \Omega$ satisfying $x, y \in S$.

It is obvious that **SCRU** implies **CRU**. All the axioms below have already appeared in the literature[14].

[14] See Hansson [8], Richter [16], Suzumura [25] and Bossert et.al [5].

Weak axiom of revealed preference (WARP) For all $x, y \in X$, if xE_Cy, then $\neg yR_Cx$.

No Ec cycle (NEcC) E_C is acyclic.

D-congruence (DC) For all $S \in \Omega$ and all $x \in S$, if xR_Cy for all $y \in S$, then $x \in C(S)$.

Strong A-congruence (SAC) For all $x, y \in X$ and all $S \in \Omega$ such that $x \in S$ and $y \in C(S)$, if either xR_Cy or there exists a finite sequence of alternatives $z_1, ..., z_t$ such that $xE_Cz_1E_C \cdots E_Cz_tE_Cy$, then $x \in C(S)$.

Definition 3 A finite sequence of sets $S_1, S_2, ..., S_t \in \Omega$ is C-related if $S_1 \cap C(S_2) \neq \emptyset$, $S_2 \cap C(S_3) \neq \emptyset$, $\ldots$, $S_{t-1} \cap C(S_t) \neq \emptyset$, and $S_t \cap C(S_1) \neq \emptyset$.

Hanson's version of the strong axiom (SA(H)) For any C-related sequence $S_1, S_2, \ldots, S_t \in \Omega$, it is true that $S_\tau \cap C(S_{\tau+1}) = C(S_\tau) \cap S_{\tau+1}$ for some $\tau \in \{1, ..., t-1\}$.

5.1 Base domain

The axiomatization depends heavily on Theorem 5 in Bossert et al. [5], which says that a choice function with a base domain is AC rational if and only if it satisfies **DC** and **NEcC**. This theorem suggests one more axiom is necessary for a choice function to be AC-IT rational. The additional axiom is **CRU**.

Theorem 3 *A choice function C with a base domain is AC-IT rational if and only if it satisfies **DC**, **NEcC**, and **CRU**.*

Proof. 'if'. According to Theorem 5 in Bossert et al., there exists a rationalization of C that is equivalent to the base relation $\succsim_C$[15]. The only thing to prove is that $\sim_C$ is transitive: Let $x \sim_C y \sim_C z$. By definition of $\sim_C$, we have $\{x, y\} = C(\{x, y\}), \{y, z\} = C(\{y, z\})$. By noting $\{x, z\} \in \Omega$, **CRU** is applied, and hence $\{x, z\} = C(\{x, z\})$, which is the desired result.

'only if'. By applying Theorem 5 in Bossert et al. again, C satisfies **DC** and **NEcC**. Let us show **CRU**. Suppose $C(S_1) \cap C(S_2) \neq \emptyset$. Let $T \in \Omega$ with $T \subset$

[15] See also Definition 2.

$C(S_1) \cap C(S_2)$. Take $x, y \in T$ arbitrary and $z \in C(S_1) \cap C(S_2)$. By letting $\succcurlyeq$ a rationalization of C, we have $x \sim z \sim y$, and hence $x \sim y$ by transitivity of $\sim$. This holds for any $x, y \in T$ so that $C(T) = G(T, \succcurlyeq) = T$. ■

We show the tightness of the axiomatization. As for **DC** and **NEcC**, it was already established (Bossert et al.[5], Examples 9 and 10 p.448). The example below illustrates that **CRU** is necessary.

Example 5 Let $X = \{x, y, z\}$, $\Omega = \Omega_B$, and let a choice function C be defined by $\{x, y\} = C(\{x, y\}), \{y, z\} = C(\{y, z\})$, and $C(\{x, z\}) = \{x\}$. Obviously C is *AC* rational so that it satisfies **DC** and **NEcC** due to Theorem 5 of Bossert et al.[5]. It is also obvious that C is not AC-IT rational and violates **CRU**.

The example below shows that Theorem 3 does not hold for an arbitrary domain.

Example 6 Let $X = \{x, y, z, w\}$, $\Omega = \{\{x, y\}, \{y, z\}, \{y, w\}, \{x, z, w\}\}$, and let a choice function C be defined by $C(\{x, y\}) = \{x, y\}$, $C(\{y, z\}) = \{y, z\}$, $C(\{y, w\}) = \{y, w\}$, and $C(\{x, z, w\}) = \{x, w\}$. C satisfies **DC**, **NEcC**, and **CRU**, but not AC-IT rational.

5.2 Arbitrary domain

The logical relationship of AC rationality with other choice axioms was established by Suzumura [25] and Bossert et al.[5]. We make its AC-IT counterpart, which is summarized in Theorem 4. Note that **SCRU** commits in all the arrows (1)-(9) in this theorem.

Theorem 4 *Let Ω be an arbitrary domain. The followings logical relationship (1)-(9) are obtained, where unnumbered arrows were already shown in the literature.*[16]

[16]Refer to Suzumura [25] and Bossert et al. [5].

$$\begin{tikzcd}
 & FR \arrow[dl, Rightarrow, "(1)"'] \arrow[drr, Rightarrow, "(2)"] & & \\
\mathbf{SA(H)+SCRU} \arrow[r, Rightarrow, "(3)"'] \arrow[d, Rightarrow, "(6)"'] & \mathbf{WARP+NEcC+SCRU} \arrow[r, Leftrightarrow, "(4)"'] \arrow[d, Rightarrow, "(7)"'] & \mathbf{SAC+SCRU} \arrow[r, Rightarrow, "(5)"'] \arrow[d, Rightarrow, "(8)"'] & AC\text{-}ITR \arrow[d, Rightarrow, "(9)"'] \\
\mathbf{SA(H)} \arrow[r, Rightarrow] & \mathbf{WARP+NEcC} \arrow[r, Rightarrow] & \mathbf{SAC} \arrow[r, Rightarrow] & ACR
\end{tikzcd}$$

Proof. (1) $FR \Longrightarrow$ **SA(H)** has already been shown in Suzumura [25]. Thus, we show $FR \Longrightarrow$ **SCRU**. Take $x, y \in X$ arbitrarily. Suppose that there is a finite sequence of alternatives $z_1, ..., z_t$ such that $xI_C z_1 I_C z_2 I_C \cdots I_C z_t I_C y$. This means that there exist $S_1, S_2, ..., S_{t+1} \in \Omega$ such that $x, z_1 \in C(S_1), z_1, z_2 \in C(S_2), ..., z_{t-1}, z_t \in C(S_t), z_t, y \in C(S_{t+1})$. This implies that $x \sim z_1 \sim z_2 \sim \cdots \sim z_t \sim y$ where $\succcurlyeq$ is a rationalization of C. Using transitivity of $\sim$, we have $x \sim y$, which together with FR completes the proof. The example below illustrates that the converse is not true.

Example 7 Let $X = \{x, y, z, w\}$, $\Omega = \{\{x, y\}, \{y, z\}, \{z, w\}, \{x, w\}\}$, and let a choice function C be defined by $C(\{x, y\}) = \{x, y\}$, $C(\{y, z\}) = \{y\}$, $C(\{z, w\}) = \{z, w\}$, and $C(\{x, w\}) = \{w\}$. It is obvious that FR is violated and that **SCRU** is trivially satisfied. There exist three C-related, $\{\{x, y\}, \{y, z\}\}, \{\{x, w\}, \{z, w\}\}$ and $\{\{x, y\}, \{y, z\}, \{z, w\}, \{x, w\}\}$. Thus we know that **SA(H)** is also satisfied.

(2) The example below illustrates that the converse is not true.

Example 8 Let $X = \{x, y, z\}$, $\Omega = \{\{x, y\}, \{y, z\}, \{x, z\}\}$, and let a choice function C be defined by $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, and $C(\{x, z\}) = \{x, z\}$.

(3) Example 8 illustrates that the converse is not true: It is obvious that **WARP**, **NEcC**, and **SCRU** are satisfied. **SA(H)** is violated: Since $\{x, y\} \cap C(\{y, z\}) = \{y\}, \{y, z\} \cap C(\{x, z\}) = \{z\}$, and $\{x, z\} \cap C(\{x, y\}) = \{x\}$, the sequence $\{\{x, y\}, \{y, z\}, \{z, x\}\}$ is C-related. However $C(\{x, y\}) \cap \{y, z\} = C(\{y, z\}) \cap \{x, z\} = \emptyset$.

(4) It suffices to show that **SAC+SCRU** $\Longrightarrow$ **WARP+NEcC**.[17]

WARP: Suppose that there exists $S \in \Omega$ such that $x \in C(S)$ and $y \in S \backslash C(S)$. Take $S' \in \Omega$ such that $x, y \in S'$. If $y \in C(S')$ then **SAC** implies $x \in C(S')$, which

[17]Example 14 of Bossert et al. (2007) shows that **SAC** $\Longrightarrow$ **WARP+NEcC** does not hold.

violates **SCRU**. Thus we have $y \notin C(S')$ which completes the proof.

NEcC: On the contrary, suppose that $x_1 E_C x_2$, $x_2 E_C x_3, \ldots, x_{t-1} E_C x_t$, and $x_t E_C x_1$. By definition of E_C, there exists a sequence of $S_1, \ldots, S_t \in \Omega$ such that $x_1 \in C(S_1)$, $x_2 \in S_1 \setminus C(S_1)$, $x_2 \in C(S_2)$, $x_3 \in S_2 \setminus C(S_2), \ldots, x_t \in C(S_t)$, and $x_1 \in S_t \setminus C(S_t)$. On the other hand, **SAC** implies $x_1 \in C(S_t)$, a contradiction.

(5) The desired AC-IT rationalization is defined with the following two binary relations $>$ and $\simeq$.

$>$: A binary relation $>$ on X is defined in the following steps: For the sake of convenience, for any $x, y \in X$, we write $x\widetilde{E}_C y$ if there exist $z_1, \ldots, z_t \in X$ such that $x E_C z_1$, $z_1 E_C z_2, \ldots, z_{t-1} E_C z_t$, and $z_t E_C y$. For any $x, y \in X$ with $x \neq y$, if $x\widetilde{E}_C y$ and $\neg y\widetilde{E}_C x$, then $x > y$. It is obvious that $>$ is well defined and transitive. We extend $>$ in the following way. Let $Y = \{y \in X : \exists x \in X \text{ s.t. } y > x \text{ or } x > y\}$. Let Y^0 be the set of minimal elements of Y with respect to $>$, that is, $Y^0 = \{y \in Y : \neg[x \in Y \text{ s.t. } y > x]\}$. We index all members in Y^0 such that $y_0^1, \ldots, y_0^m$ and define $y_0^j > y_0^{j'}$ if $j > j'$. Next let Y^1 be the set of minimal elements of $Y \setminus Y^0$ with respect to $>$, that is, $Y^1 = \{y \in Y \setminus Y^0 : \neg[x \in Y \setminus Y^0 \text{ s.t. } y > x]\}$. We define $>$ on Y^1 in the same way as on Y^0 and let $y^1 > y^0$ for all $y^1 \in Y^1$ and all $y^0 \in Y^0$. Let Y^2 be the set of minimal elements of $Y \setminus (Y^0 \cup Y^1)$ with respect to $>$, that is, $Y^2 = \{y \in Y \setminus (Y^0 \cup Y^1) : \neg[x \in Y \setminus Y^0 \cup Y^1 \text{ s.t. } y > x]\}$. We define $>$ on Y^2 in the same way as on Y^0 and Y^1 and let $y^2 > y^0$, $y^2 > y^1$ for all $y^2 \in Y^2$, all $y^0 \in Y^0$, and all $y^1 \in Y^1$. We define $Y^3, \ldots, Y^k$ and $>$ on $Y^0 \cup \cdots \cup Y^k$ in the same way. All alternatives in Y are used up at Y^k. Finally we set $x > y$ for all $x \in X \setminus Y$ and all $y \in Y$; $x > x'$ or $x' > x$ for any $x, x' \in X \setminus Y$ with $x \neq x'$; and $>$ is transitive on $X \setminus Y$. By definition, $>$ is transitive and either $x > y$ or $y > x$ holds for any distinct x and y.

$\simeq$: Define the binary relation $\simeq$ on X such that for any $x, y \in X, x \simeq y$ if and only if there exist $z_1, ..., z_t \in X$ such that $x I_C z_1 I_C \cdots I_C z_t I_C y$. Note that $\simeq$ is an equivalence relation on $\{x \in X : x \in C(S)\ \exists S \in \Omega\}$. A rationalization $\succsim$ of C is defined as follows: For any $x, y \in X$,

$$\begin{cases} x \sim y & \text{if } x \simeq y \text{ or } x = y \\ x \succ y \Longleftrightarrow x > y & \text{otherwise} \end{cases}$$

By construction, $\succcurlyeq$ is complete and AC-IT. We prove that $\succcurlyeq$ is a rationalization of C.

$C(S) \subset G(S, \succcurlyeq)$: Take $x \in C(S)$ and $y \in S$ arbitrarily. If $y \in C(S)$, then $xI_C y$, that is, $x \simeq y$ and hence $x \sim y$. Next suppose $y \in S \backslash C(S)$. By **SCRU**, $x \simeq y$ does not hold. On the other hand $x\widetilde{E}_C y$ holds true. If $y\widetilde{E}_C x$, then **SAC** requires $y \in C(S)$ which is a contradiction. Thus we have $x\widetilde{E}_C y$ and $\neg\, y\widetilde{E}_C x$, that is, $x > y$. By definition of $\succcurlyeq$, we have $x \succ y$, which is the desired result.

$G(S, \succcurlyeq) \subset C(S)$: Take $x \in G(S, \succcurlyeq)$ and $y \in C(S)$ arbitrarily. If $x \simeq y$ holds, then **SCRU** requires $x \in C(S)$, which is the desired result. Next suppose $x \simeq y$ does not hold. Then we have $x \in S \backslash C(S)$, and hence $y\widetilde{E}_C x$. If $x\widetilde{E}_C y$ does not hold, then $y > x$, that is, $y \succ x$, which contradicts $x \in G(S, \succcurlyeq)$. Thus $x\widetilde{E}_C y$ holds. **SAC** requires $x \in C(S)$, which is a contradiction.

The example below illustrates that the converse is not true.

Example 9 Let $X = \{x, y, z\}$, $\Omega = \{\{x, y\}, \{x, z\}, \{x, y, z\}\}$, and let a choice function C be defined by $C(\{x, y\}) = \{x, y\}, C(\{x, z\}) = \{z\}$, and $C(\{x, y, z\}) = \{y\}$. It is obvious that C is AC-IT rational, but violates **SCRU** and **SAC**. **SAC** does not hold because $x \in \{x, y, z\}, y \in C(\{x, y, z\}), x \in C(\{x, y\})$ and $x \notin C(\{x, y, z\})$.

(6) The example below illustrates that the converse is not true.

Example 10 Let $X = \{x, y, z\}$, $\Omega = \{\{x, y\}, \{y, z\}, \{x, z\}\}$, and let a choice function C be defined by $C(\{x, y\}) = \{x, y\}$, $C(\{y, z\}) = \{y, z\}$, and $C(\{x, z\}) = \{x\}$. C satisfies **SA(H)** but not **SCRU**. Note that there exist three C-related: $\{\{x, y\}, \{y, z\}\}, \{\{x, y\}, \{x, z\}\}$, and $\{\{x, y\}, \{x, z\}, \{y, z\}\}$.

(7) Example 10 illustrates that the converse is not true.

(8) If the converse is true, then **WARP+NEcC** $\Longleftrightarrow$ **SAC**, a contradiction.

(9) Example 10 illustrates that the converse is not true. $\blacksquare$

6 AC-IT rational choice and solution concepts in games

Let us assume $\Omega = \Omega_F$ again.

Definition 4 Let C be a choice function on Ω_F and $\succcurlyeq$ be a complete binary relation on X.

(i) C is the $\succcurlyeq$-*core* if $C(S) = G(S, \succcurlyeq)$ for all $S \in \Omega_F$.

(ii) C is the *strict* $\succcurlyeq$-*core* if for all $S \in \Omega_F$,

(a) $C(S) = G(S, \succcurlyeq)$; and

(b) for all $x \in C(S)$ and all $y \in S\setminus C(S)$, $x \sim y$ implies that there exists $z \in S$ such that $x \succ z$ and $z \succ y$.

(iii) C is the $\succcurlyeq$-*stable set*[18] if for all $S \in \Omega_F$,

(a) (internal stability) $x \sim y$ for all $x, y \in C(S)$; and

(b) (external stability) for all $y \in S \backslash C(S)$, there exists $x \in C(S)$ such that $x \succ y$.

It would be easier to understand the definition if we note that $x \succ y$ reads 'x dominates y' and that $x \sim y$ reads 'x and y are not dominated by each other.'

Definition 5 Let C be a choice function. For any $S \in \Omega_F$, an *indifference closure* of $C(S)$, denoted by $\widehat{C(S)}$, is defined as follows:

$$\widehat{C(S)} = \bigcup_{x \in C(S)} I(x) \cap S,$$

where $I(x) = \{y \in X | x \sim_C y\}$.

It is straightforward from the definition that $C(S) \subset \widehat{C(S)}$ for all $S \in \Omega$. Note that $\widehat{C}$ is a choice function.

Theorem 5 *Let C be a choice function. Then, the following three statements are equivalent:*

(i) C is AC-IT rational.

(ii) C is the strict $\succcurlyeq_C$-core.

(iii) C is the $\succcurlyeq_C$-core and $\widehat{C}$ is the $\succcurlyeq_C$-stable set.

[18] Von Neumann and Morgenstern [29] showed that if $\succcurlyeq$ is acyclic, then there exists a unique $\succcurlyeq$-stable set.

Proof. (ii)$\Longrightarrow$(i): Suppose that C is the strict $\succcurlyeq_C$-core. By definition, C is the $\succcurlyeq_C$-core, which means that $\succcurlyeq_C$ is a rationalization of C. Next, we show that $\succcurlyeq_C$ is AC-IT. By Theorem 2.10 of Suzumura p36 [25], if C is $\succcurlyeq_C$-core, then $\succcurlyeq_C$ is acyclic. Thus, it suffices to show that $\succcurlyeq_C$ is IT. Suppose that $\succcurlyeq_C$is not IT. Without loss of generality, we assume $x \sim_C y$, $y \sim_C z$ and $x \succ_C z$ for some $x, y, z \in X$. Since $\succcurlyeq_C$ rationalizes C, we have $C(\{x, y, z\}) = \{x, y\}$. Since C is the strict $\succcurlyeq_C$-core, $y \sim_C z$ implies that $y \succ_C x$ and $x \succ_C z$, which contradicts $x \sim_C y$.

(i) $\Longrightarrow$ (iii): Suppose that C is rationalizable with some AC-IT relation $\succcurlyeq$. Then, it is easy to see that the base relation $\succcurlyeq_C$ is also AC-IT. Since $\succcurlyeq_C$ is acyclic, it is clear that C is the $\succcurlyeq_C$-core. We show that $\widehat{C}$ is the $\succcurlyeq_C$-stable set. At first, let us prove the internal stability of $\widehat{C(S)}$ for any $S \in \Omega_F$. For any $x, y \in \widehat{C(S)}$, there exist $z, w \in C(S)$ such that $x \sim_C z$ and $y \sim_C w$. Since $z \sim_C w$, IT of $\succcurlyeq_C$ implies $x \sim_C y$. Thus, $\widehat{C(S)}$ is internally stable. Next, we show that $\widehat{C(S)}$ is externally stable. Let $y \in S \setminus \widehat{C(S)}$. Since C is the $\succcurlyeq_C$-core, $x \succcurlyeq_C y$ for all $x \in C(S)$. It follows from $y \notin \widehat{C(S)}$ that $x \succ_C y$ for all $x \in C(S) \subset \widehat{C(S)}$. This means that $\widehat{C(S)}$ is externally stable.

(iii) $\Longrightarrow$ (ii): Suppose that C is the $\succcurlyeq_C$-core and $\widehat{C}$ is the $\succcurlyeq_C$-stable set. Then, we show that C is the strict $\succcurlyeq_C$-core. For this, it suffices to prove (ii)-(b). Let $S \in \Omega_F$, $x \in C(S)$, and $y \in S \setminus C(S)$ with $x \sim_C y$. Note that $y \in \widehat{C(S)}$. Since $y \in S \setminus C(S)$, there exists $z \in S$[19] such that $z \succ_C y$. Since $x \in C(S)$, it holds that $x \succcurlyeq z$. If $x \sim_C z$, then $z \in \widehat{C(S)}$, and hence the internal stability of $\widehat{C(S)}$ implies $y \sim_C z$. That is a contradiction, so we obtain $x \succ_C z$. Therefore, we conclude that C is a strict $\succcurlyeq_C$-core. ■

Bandyopadhyay and Sengupta [3] showed that a choice function is rationalizable with a quasi-transitive relation if and only if every choice set of the function is the stable set. Theorem 5 and this result make the difference between AC-IT rationality and quasi-transitivity rationality clear.

[19]In particular, $z \in \widehat{C(S)}$ holds here. Caution: Not $z \in C(S)$!

7 AC-IT collective choice rules

Let $N = \{1, ..., n\}$ be a set of individuals. A preference of every individual is represented by a complete and transitive binary relation $\succcurlyeq$ on X. A preference profile is denoted by $\pi = (\succcurlyeq_1, \ldots, \succcurlyeq_n)$. Let Π be the set of preference profiles. For any $\pi \in \Pi$ and any $x, y \in X$, $\pi_{\{x,y\}}$ is the restriction of π on $\{x, y\}$, that is, the preference profile over $\{x, y\}$. A *collective choice rule* is a map $F : \Omega \times \Pi \to \Omega_F$ such that $F(\cdot, \pi)$ is a choice function for any $\pi \in \Pi$.

Definition 6 A collective choice rule F satisfies

(i) *Pareto* (P) if for all $x, y \in X$ and all $\pi \in \Pi$, $x \succ_i y$ for all $i \in N$ implies $F(\{x, y\}, \pi) = \{x\}$.

(ii) *independence of irrelevant alternatives* (IIA) if for all $x, y \in X$ and all $\pi, \pi' \in \Pi$, $\pi\{x,y\} = \pi'_{\{x,y\}}$ implies $F(\{x, y\}, \pi) = F(\{x, y\}, \pi')$.

Definition 7 $i \in N$ is a *vetoer* in F if for all $x, y \in X$ and all $\pi \in \Pi$, $x \succ_i y$ implies $x \in F(\{x, y\}, \pi)$.

Theorem 6 *Let F be a collective choice rule defined on a base domain and satisfying **P** and **IIA**. Suppose that for all $\pi \in \Pi$, the base relation of $F(\cdot, \pi)$ is AC-IT. If $|X| \geq 4$, then there exists a unique vetoer $i \in N$ in F.*

Proof. Let a mapping that associates each $\pi \in \Pi$ with a binary relation π_F on X be such that:

$$\text{For any } \pi \in \Pi \text{ and any } x, y \in X,\ x\pi_F y \Longleftrightarrow x \in F(\{x, y\}, \pi).$$

This mapping is a social choice rule of Iritani, Kamo and Nagahisa [11], and hence their vetoer theorem is applied and the proof completes. ■

Applying Lemmas 1 and 2 to Theorem 6, we obtain the following theorem.

Theorem 7 *Let F be a collective choice rule defined on the full domain and satisfying **P** and **IIA**. Suppose that for all $\pi \in \Pi$, $F(\cdot, \pi)$ satisfies **CA** and **RU**. If $|X| \geq 4$, then there exists a unique vetoer $i \in N$ in F.*

8 Conclusion

It is somehow amazing that although a significant amount of literature has studied rational choice functions, no literature has dealt with AC-IT rational choice functions. The table in Section 4 however reveals that there exists a meaningful logical relationship between axiomatizations of *FR*, *QTR*, AC and AC-IT rational choice functions. The study of AC-IT rational choice was therefore probably a missing piece of the jigsaw puzzle in the study of rational choice functions.

9 Appendix

9.1 Properties of RU

Definition 8 We say that $S \in \Omega_F$ is $\sim_C$ *connected* if for any $x, y \in S$, there are a finite number of alternatives $z_1, ..., z_t \in S$ such that $x \sim_C z_1 \sim_C \cdots \sim_C z_t \sim_C y$.

Definition 9 We say that $S \in \Omega_F$ is a recursive set if $C(S) = S$.

Recursivity condition (RC) For any $S \in \Omega_F$, if S is $\sim_C$ connected, S is a *recursive set.*

If a choice function satisfies **CA**, then the converse of **RC** holds with any two alternatives being directly connected. Otherwise there is some recursive set that is not $\sim_C$ connected. Example 1 illustrates this.

Theorem 8 *The following holds:*

*(i) For any choice function C satisfying **CA**, if it satisfies **RC**, then it satisfies **RU**; and*

*(ii) If a choice function C satisfies **RU**, then it satisfies **RC**.*

Proof. (i): Let $C(S_1) \cap C(S_2) \neq \emptyset$. It suffices to show $C(S_1) \cup C(S_2)$ is $\sim_C$ connected. (i) of Lemma 2 shows

$$C(S_1) \subset G(S_1, \succcurlyeq_C) \text{ and } C(S_2) \subset G(S_2, \succcurlyeq_C) \tag{5}$$

For any $x, y \in C(S_1) \cup C(S_2)$, if x and y belong to $C(S_1)$ [resp.$C(S_2)$], then (5) implies $x \sim_C y$. If $x \in C(S_1)$ and $y \in C(S_2)$, then picking $z \in C(S_1) \cap C(S_2)$ arbitrarily, (5) implies $x \sim_C z \sim_C y$. Thus $C(S_1) \cup C(S_2)$ is $\sim_C$ connected.

(ii): Suppose that S is $\sim_C$ connected. The proof goes on with induction of the number of the elements of S. It is obvious when the number is 1 or 2. Assuming that (ii) is true when the number is less than or equal to $k - 1 (\geqslant 2)$, we consider the case of k. Take an element e in S arbitrarily. We call e *Erdös*. The idea of the following proof is based on Erdös number. Let e have Erdös number 0. Let any element $x \neq e$ in S that directly connects with e, that is, $e \sim_C x$, have Erdös number 1. Let any element $y \in S$ that cannot directly connect with e but connect with some point of Erdös number 1 have Erdös number 2, and so on. By noting that S is $\sim_C$ connected, there are elements with the maximum Erdös number. Take one of them, denoted by x_k. Let us show $S \backslash \{x_k\}$ is $\sim_C$ connected. Note that any two points $x, y \in S \backslash \{x_k\}$ are connected via e, that is,

$$x \sim_C x_1 \sim_C \cdots \sim_C x_t \sim_C e \sim_C y_q \sim_C \cdots \sim_C y_1 \sim_C y.$$

Note also that x_k cannot join this sequence: If so, it contradicts the fact that x_k has the maximum Erdös number. This completes the proof of $\sim_C$ connectedness of $S \backslash \{x_k\}$. By inductive hypothesis, $S \backslash \{x_k\}$ is a recursive set, so that $S \setminus \{x_k\} = C(S \setminus \{x_k\})$. Let x_{k-1} be such that $x_{k-1} \sim_C x_k$, we have $\{x_{k-1}, x_k\} = C(\{x_{k-1}, x_k\})$. **RU** is applied, so that $C(S \backslash \{x_k\}) \cup C(\{x_{k-1}, x_k\}) = C(C(S \setminus \{x_k\}) \cup C(\{x_{k-1}, x_k\}))$, which means $S = C(S)$, which is the desired result. ■

(i) is not true without **CA**; replacing $C(\{x, y, z\}) = \{x, y, z\}$ in Example 1 with $C(\{x, y, z\}) = \{x, y\}$, we have an example that illustrates this.

9.2 Logical relationship between RU and other axioms

RU logically relates to the following axioms.

Dual Chernoff's axiom (DCA) For all $S_1, S_2 \in \Omega_F$, if $S_1 \subset S_2$ and $S_1 \cap C(S_2) \neq \emptyset$, then $S_1 \cap C(S_2) \supset C(S_1)$.

Stability (ST) For all $S \in \Omega_F$, $C(S) = C(C(S))$[20].

[20]The terminology follows Suzumura[25]. This is called Idempotence in mathematics.

ST is weaker than **RU**. **DCA** is a dual concept of **CA**. **CA**+**DCA** is called Arrow's axiom, known as a necessary and sufficient condition for a choice function to be full rational.[21]

Theorem 9 *If a choice function C satisfies **DCA** and **ST**, then it satisfies **RU**.*

Proof. Suppose $C(S_1) \cap C(S_2) \neq \emptyset$. The key to the proof is to apply **DCA** between $C(S_1)$ and $C(S_1) \cup C(S_2)$, and between $C(S_2)$ and $C(S_1) \cup C(S_2)$.

For two sets, $C(S_1) \cap C(C(S_1) \cup C(S_2))$ and $C(S_2) \cap C(C(S_1) \cup C(S_2))$, we need to consider the following three cases: both are nonempty (case 1), one is nonempty and the other is empty (case 2), and both are empty (case 3). For case 1, we can apply **DCA** and **ST** twice, so that we have $C(S_1) \cap C(C(S_1) \cup C(S_2)) \supset C(S_1)$, and $C(S_2) \cap C(C(S_1) \cup C(S_2)) \supset C(S_2)$. Thus $C(C(S_1) \cup C(S_2)) \supset C(S_1) \cup C(S_2)$, which is the desired result. For case 2, if $C(S_1) \cap C(C(S_1) \cup C(S_2)) = \emptyset$, then

$$C(S_2) - C(S_1) \quad \overset{\substack{C(S_1)\cap C(C(S_1)\cup C(S_2))=\emptyset \\ \downarrow}}{\supset} \quad C(C(S_1) \cup C(S_2)) \quad \overset{\substack{\textbf{DCA+ST} \\ \downarrow}}{\supset} \quad C(S_2),$$

which contradicts the fact that $C(S_1)$ and $C(S_2)$ have a nonempty intersection. The case of $C(S_2) \cap C(C(S_1) \cup C(S_2)) = \emptyset$ results in contradiction as well. For Case 3, we have $C(C(S_1) \cup C(S_2)) = \emptyset$, a contradiction. ■

Example 3 shows that the converse of Theorem 9, **RU** $\Longrightarrow$ **DCA**, is not true.[22] Two examples below illustrate that if either **DCA** or **ST** is lacked, Theorem 9 does not hold.

Example 11 Let $X = \{x, y, z\}$ and let a choice function C be defined by $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, $C(\{x, z\}) = \{x\}$, and $C(\{x, y, z\}) = \{x, y\}$. C satisfies **DCA**, but fails to satisfy **RU** as well as **ST**.

Example 12 Let $X = \{x, y, z\}$ and let a choice function C be defined by $C(\{x, y\}) = \{x, y\}$, $C(\{y, z\}) = \{y, z\}$, $C(\{x, z\}) = \{x, z\}$, and $C(\{x, y, z\}) = \{x, y\}$. C satisfies **ST**, but fails to satisfy **RU** as well as **DCA**.

[21] Refer to Arrow [2] and Sen [20], [21] for more details.
[22] **RU** $\Longrightarrow$ **ST** is always true.

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