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Hiroyuki TAMURA

文部科学省私立大学社会連携研究推進拠点
関西大学政策グリッドコンピューティング実験センター

Policy Grid Computing Laboratory,
Kansai University
Suita, Osaka 564-8680 Japan
URL : https://www.pglab.kansai-u.ac.jp/
e-mail : pglab@jm.kansai-u.ac.jp
tel. 06-6368-1228
fax. 06-6330-3304

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Methodology of Decision Analysis Under Risk and/or Uncertainty with Application to Public Sectors

Hiroyuki TAMURA[1]

Abstract

In this paper, some behavioral (or descriptive) models of individual decision making under risk and/or uncertainty are discussed. Firstly, a model to explain the violations of expected utility hypothesis is described. In this model outcome-dependent, non-additive probabilities are introduced in a measurable value function under risk where probability of each event occurring is known. The effective application of this approach to the public sector is shown in modeling risks of extreme events with low probability and high outcome. Next, a measurable value function under uncertainty is also described where basic probability of a set of event is known but occurrence probability of each event is not known. Potential applicability to evaluating a global warming problem is mentioned. As a special case of the measurable value function under uncertainty an extended Kahneman-Tversky model of prospect theory under uncertainty (PTU) is described. An application of PTU in evaluating the sense of security provided by nursing care robots is described.

Keywords: Individual decision making; Behavioral (descriptive) model; Utility theory; Expected utility paradox; Prospect theory under uncertainty.

[1] Faculty of Engineering Science, Kansai University

1. INTRODUCTION

A normative (prescriptive) model of decision making prescribes optimal behavior of how decisions should be made. On the other hand, a descriptive model is concerned with understanding how people actually behave when making decisions.

The expected utility model has been widely used as a normative model of decision analysis under risk for modeling individual decision making. However, various paradoxes (Allais and Hagen, 1979; Ellsberg, 1961) have been reported for the expected utility model, and it is argued that the expected utility model is not an adequate behavioral (descriptive) model. As a model to explain the violations of the expected utility hypothesis for individual decision making, outcome-dependent, non-additive probabilities are introduced in a measurable value function under risk where the probability of each event occurring is postulated to be known. The effective application of this approach to the public sector is mentioned in modeling risks of extreme events with low probability and high outcome.

A measurable value function under uncertainty is also described where the basic probability of a set of event is known but the probability of each event occurring is not known. It is shown that the Ellsberg (1961) paradox is consistently resolved by using this model. This is a paradox in decision theory and experimental economics in which people's choices violate the expected utility hypothesis because of their tendency for ambiguity aversion. Potential applicability to evaluating a global warming problem is mentioned.

As a special case of a measurable value function under uncertainty, prospect theory under uncertainty (PTU) is described as an extended Kahneman and Tversky (1979) model of prospect theory. An application of PTU to evaluating the sense of security provided by nursing care robots is described.

2. EXPECTED UTILITY MODEL

Utility functions could provide a means of modeling value judgment of a decision maker quantitatively. The scientific approach for value judgment has been discussed rigorously in the area of economics. People get a feeling of psychological satisfaction by consuming economic goods, by receiving service and so forth. This degree of satisfaction obtained is called “utility.” This concept plays a fundamental role in the theory of consumers’ behavior.

Let x_1 and x_2 be the amount of goods A and B consumed, respectively, $u(x_1, x_2)$ be the corresponding value of the consumer’s utility function, p_1 and p_2 be the price of a unit amount of goods A and B, respectively, and b be the budget. Then, the consumer may want to

$$\text{maximize} \quad u(x_1, x_2) \tag{1}$$

$$\text{subject to} \quad p_1 x_1 + p_2 x_2 \le b. \tag{2}$$

That is, the consumer behavior has been explained in such a way that they would act to maximize their own utility, Eq. (1), subject to the budget constraint, Eq. (2).

In order to derive the equilibrium condition of a consumer’s behavior, an ordinal utility function would be enough to evaluate it, but, for obtaining a preferred solution for a multiple criteria decision-making problem, we need a cardinal utility function. Furthermore, since, in a decision making problem under risk, the outcome would be obtained under some probability distribution, we need to evaluate the so-called expected utility (Hammond, 1998). For this we need to provide a cardinal utility function. Von Neumann and Morgenstern (1947) first developed axioms such that the expected utility hypotheses for the decision making problem under risk are meaningful.

Let X be a set of outcomes, $u: X \rightarrow \mathrm{Re}$ be a cardinal utility function, then the expected utility with respect to the probability on X is

$$E(u, p) = \sum_{x \in X} p(x) u(x) \tag{3}$$

which is called the expected utility. Let $P = \{p_1, p_2, \ldots\}$ be the set of probabilities on X. Then the following theorem provides the existence and uniqueness of a cardinal utility function (von Neumann and Morgenstern, 1947).

Theorem: Existence and uniqueness of a cardinal utility function

Let P be the set of all probabilities on X, $(P, \succeq)$ be preference structure on P, then for any $p, q \in P$, the necessary and sufficient condition of the existence of a cardinal utility function $u: X \rightarrow \mathrm{Re}$ such that

$$p \succeq q \Leftrightarrow E(u, p) \geq E(u, q), \quad \forall p, q \in P \tag{4}$$

is given as follows:

NM1: $(P, \succeq)$ is weak order.

NM2: $p \succ q \Rightarrow \alpha p + (1-\alpha) r \succ \alpha q + (1-\alpha) r, \quad \forall r \in P, \quad \alpha \in (0,1)$

NM3: $p \succ q \succ r \Rightarrow \alpha p + (1-\alpha) r \succ q \succ \beta p + (1-\beta) r$, for some $\alpha, \beta \in (0,1)$

Furthermore, such u is unique within the positive linear transformation (there exist h and $k > 0$ such that $u' = h + ku$) and is called a von Neumann- Morgenstern utility function.

The expected utility model based on the expected utility hypothesis is useful as a normative model, that is, to find a decision to be made. However, since various paradoxes (Allais and Hagen, 1979; Ellsberg, 1961) have been reported for the expected utility model, it is argued that the expected utility model is not an adequate behavioral (descriptive) model. Actually, there exist many phenomena that violate the expected utility hypothesis such as the Allais paradox (Allais and Hagen, 1979) and the Ellsberg (1961) paradox. In the next section we describe a generalized model for a measurable value function under risk and a measurable value function under uncertainty (Tamura, 2005) to overcome the difficulty of the expected utility paradoxes.

3. BEHAVIORAL MODELS TO RESOLVE EXPECTED UTILITY PARADOXES

3.1 Measurable value function under risk

The expected utility model has been widely used as a normative model of decision analysis under risk. However, various paradoxes have been reported for the expected utility model, and it is argued that the expected utility model is not an adequate descriptive model.

In this section a descriptive extension of the expected utility model to account for various paradoxes is shown using the concept of strength of preference (Tamura, 2005). Let X be a set of all outcomes, $x \in X$, and A be a set of all risky alternatives; a prospect (risky alternative) $\ell \in A$ is written as

$$\ell = (x_1, x_2, \ldots, x_n; p_1, p_2, \ldots, p_n) \tag{5}$$

that yields outcome $x_i \in X$ with probability $p_i, i = 1,2,\ldots,n,$ where $\sum p_i = 1$.

Let A^* be a nonempty subset of $A \times A$, and $\succeq$ and $\succeq^*$ be binary relations on A and A^*, respectively. Relation $\succeq$ could also be a binary relation on *X*. We interpret $\ell_1 \succeq \ell_2$ $(\ell_1, \ell_2 \in A)$ to mean that ℓ_1 is preferred than or indifferent to ℓ_2, and $\ell_1\ell_2 \succeq^* \ell_3\ell_4$ $(\ell_1, \ell_2, \ell_3, \ell_4 \in A)$ to mean that the strength of preference for ℓ_1 over ℓ_2 is greater than or equal to the strength of preference for ℓ_3 over ℓ_4.

We postulate that $(A, A^*, \succeq^*)$ takes a positive difference structure that is based on the axioms described by Kranz et al. [7]. The axioms imply that there exists a real-valued function *F* on *A* such that for all $\ell_1, \ell_2, \ell_3, \ell_4 \in A$, if $\ell_1 \succeq \ell_2$ and $\ell_3 \succeq \ell_4$, then

$$\ell_1\ell_2 \succeq^* \ell_3\ell_4 \;\Leftrightarrow\; F(\ell_1) - F(\ell_2) \geq F(\ell_3) - F(\ell_4) \tag{6}$$

Since *F* is unique up to a positive linear transformation, it is a cardinal function. It is natural to hold for $\ell_1, \ell_2, \ell_3 \in A$ that

$$\ell_1\ell_3 \succeq^* \ell_2\ell_3 \;\Leftrightarrow\; \ell_1 \succeq \ell_2 . \tag{7}$$

Then from Eq. (6) we obtain

$$\ell_1 \succeq \ell_2 \;\Leftrightarrow\; F(\ell_1) \geq F(\ell_2). \tag{8}$$

Thus, *F* is a value function on *A* and, in view of Eq. (6), it is a measurable value function.

We assume that the decision maker will try to maximize the value (or utility) of a prospect (risky alternative) $\ell \in A$, which is given by the general form as follows:

$$\max_{\ell \in A} F(\ell) = \max_{\ell \in A} \sum_i f(x_i, p_i) \tag{9}$$

where $f(x, p)$ denotes the value (strength of preference) for an outcome *x* which comes out with probability *p*. This function is called the *measurable value function under risk*. The main objectives here are to give an appropriate decomposition and interpretation of $f(x, p)$ and to explore its descriptive implications to account for the various paradoxes.

The model of Eq. (9) is reduced to the expected utility form by setting

$$f(x, p) = pu(x) \tag{10}$$

when $u(x)$ is regarded as a von Neumann-Morgenstern utility function. The prospect theory of Kahneman and Tversky (1979) is obtained by setting

$$f(x, p) = \pi(p)v(x) \tag{11}$$

where $\pi(p)$ denotes a weighting function for probability and $v(x)$ a value function for outcome. In this model the value of each outcome is multiplied by a decision weight for probability (not by probability itself).

Extending this Kahneman-Tversky model we obtain a decomposition form

$$f(x, p) = w(p \mid x)v(x) \tag{12}$$

where

$$w(p \mid x) \equiv \frac{f(x, p)}{f(x,1)} \tag{13a}$$

$$v(x) \equiv v(x \mid 1) \tag{13b}$$

$$v(x \mid p) \equiv \frac{f(x, p)}{f(x^*, p)} \tag{13c}$$

and x^* denotes the best outcome. The expected utility model, Eq. (10), and Kahneman-Tversky model, Eq. (11), are included in our model, Eq. (12), as a special case. Eq. (13b) implies that $v(x)$ denotes a measurable value function under certainty. Therefore, our model, Eq. (12), also includes Dyer and Sarin's (1979) model as a special case.

The model of Eq. (12) could also be written as

$$f(x, p) = w(p)v(x \mid p) \tag{14}$$

where

$$w(p) \equiv w(p \mid x^*). \tag{15}$$

We assume that

$$f(x,0) = 0, \quad \forall x \in X \tag{16a}$$

$$f\left(x^R, p\right) = 0, \quad \forall p \in [0,1] \tag{16b}$$

where $x^R \in X$ denotes the reference point (e.g. status quo). The better region on X compared with x^R is called the gain domain and the worse region the loss domain. We also assume that $f(x, p) \geq 0$ in the gain domain and $f(x, p) < 0$ in the loss domain.

It will be shown that the conditional weighting function $w(p \mid x)$, that is an outcome-dependent, non-additive probability, describes the strength of preference for probability under the given conditional level of outcome, and $v(x \mid p)$ describes the strength of preference for outcome under the given conditional level of probability.

In interpreting the descriptive model $f(x, p)$ we need to interpret F such that Eq. (6) holds. For all $x_1, x_2, x_3, x_4 \in X$, $\alpha \in [0,1]$ and $y \in X$ such that $x_1 \succeq x_2 \succeq x_3 \succeq x_4$, we consider four alternatives:

$$\ell_1 = (x_1, y; \alpha, 1-\alpha), \quad \ell_2 = (x_2, y; \alpha, 1-\alpha), \quad \ell_3 = (x_3, y; \alpha, 1-\alpha), \quad \ell_4 = (x_4, y; \alpha, 1-\alpha). \tag{17}$$

In this case we obtain

$$\ell_1 \ell_2 \succeq^* \ell_3 \ell_4 \quad \Leftrightarrow \quad f(x_1, \alpha) - f(x_2, \alpha) \geq f(x_3, \alpha) - f(x_4, \alpha) \tag{18a}$$

$$\Leftrightarrow \quad v(x_1 \mid \alpha) - v(x_2 \mid \alpha) \geq v(x_3 \mid \alpha) - v(x_4 \mid \alpha). \tag{18b}$$

Therefore, the value function $v(x \mid p)$ defined by Eq. (13c) represents the strength of preference for the four risky alternatives in Eq. (17).

On the other hand, for all $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0,1]$, $x \in X$ and $x^R \in X$, we consider four alternatives:

$$\ell_1' = \left(x, x^R; \alpha_1, 1-\alpha_1\right), \quad \ell_2' = \left(x, x^R; \alpha_2, 1-\alpha_2\right), \ell_3' = \left(x, x^R; \alpha_3, 1-\alpha_3\right), \quad \ell_4' = \left(x, x^R; \alpha_4, 1-\alpha_4\right), \tag{19}$$

then we obtain

$$\ell_1' \ell_2' \succeq^* \ell_3' \ell_4' \quad \Leftrightarrow \quad f(x, \alpha_1) - f(x, \alpha_2) \geq f(x, \alpha_3) - f(x, \alpha_4) \tag{20a}$$

$$\Leftrightarrow \quad w(\alpha_1 \mid x) - w(\alpha_2 \mid x) \geq w(\alpha_3, x) - w(\alpha_4, x). \tag{20b}$$

Therefore, the weighting function defined by Eq. (13a) represents the strength of preference for the four risky alternatives in Eq. (19).

The above discussion asserts that the descriptive model $f(x, p)$ represents the measurable value function under risk to evaluate the outcome $x \in X$ that comes out with probability *p*. The Kahneman-Tversky model of Eq. (11) could explain a so-called certainty effect to resolve the Allais paradox. Our descriptive model $f(x, p)$ could also resolve the Allais paradox.

It is well known that the expected utility model is not an appropriate model for modeling extreme events with low probability and high outcome. In Tamura, *et al*. (2000) it is shown that our descriptive model could resolve such paradox in application to the public sector.

3.2 Measurable value function under uncertainty

In this section we deal with the case where the probability of occurrence for each event is unknown. When we describe the degree of ignorance and uncertainty by the basic probability of Dempster and Shafer (1976) theory, the problem is how to represent the value of a set element in constructing a measurable value function under uncertainty based on this concept.

In the Dempster-Shafer theory of probability let $\mu(A_i)$ be basic probability which could be assigned by any subset A_i of Θ, where Θ denotes a set containing every possible element. The basic probability $\mu(A_i)$ can be regarded as a semimobile probability mass. Let $\Lambda = 2^{\Theta}$ be a set containing every subset of Θ. Then the basic probability $\mu(A_i)$ is defined on Λ and takes a value contained in [0,1]. When $\mu(A_i)$>0, A_i is called the focal element or the set element and the following conditions hold:

$$\mu(\phi) = 0 , \qquad \sum_{A_i \in A} \mu(A_i) = 1 \tag{21}$$

where ϕ denotes an empty set.

Let the value function under uncertainty based on this basic probability be

$$f^*(B, \mu) = w'(\mu) v^*(B \mid \mu) \tag{22}$$

where *B* denotes a set element, μ denotes the basic probability, w' denotes the weighting function for the basic probability, and *v** denotes the value function with respect to a set element. The set element *B* is a subset of $\Lambda = 2^{\Theta}$. Eq. (22) is an extended version of the value function, Eq. (14), where an element is extended to a set element and the Bayes' probabilty is extended to the Dempster-Shafer basic probability.

For identifying *v**, we need to find the preference relations among set elements, which is not an easy task. If the number of elements contained in the set Θ is getting larger, and the set element *B* contains a considerable number of elements, it is not practical to find *v** as a function of *B*. To cope with this difficulty we could use some appropriate axiom of dominance as follows:.

Axiom of Dominance 1.

In the set element *B* let the worst outcome be m_B and the best outcome be M_B. For any $B1, B2 \subset \Lambda = 2^{\Theta}$

$$m_{B1} \preceq m_{B2},\ M_{B1} \preceq M_{B2} \ \Rightarrow\ B1 \preceq B2 \tag{23}$$

and

$$m_{B1} \sim m_{B2},\ M_{B1} \sim M_{B2} \ \Rightarrow\ B1 \sim B2. \tag{24}$$

Our descriptive model $f^*(B, \mu)$ could resolve the Ellsberg paradox by restricting a set element *B* to

$$\Omega = \{(m, M) \in \Theta \times \Theta : m \preceq M\} \tag{25}$$

where *m* and *M* denote the worst and the best outcome in the set element *B*, respectively. In this case Eq. (22) is reduced to

$$f^*(\Omega,\mu) = w'(\mu)\, v^*(\Omega \mid \mu). \tag{26}$$

Suppose we look at an index of optimism $\alpha(m,M)$ such that the following two alternatives are indifferent (Jaffray 1988).

Alternative 1. One can receive *m* for the worst case and *M* for the best case. There exists no other information.

Alternative 2. One receives *M* with probability $\alpha(m,M)$ and receives *m* with probability $1-\alpha(m,M)$, where $0 < \alpha(m,M) < 1$.

If one is quite optimistic, $\alpha(m,M)$ becomes nearly equal to 1, or if one is quite pessimistic, $\alpha(m,M)$ becomes nearly equal to zero. If we incorporate this optimism index $\alpha(m,M)$ in Eq. (26), the value function is obtained as

$$v^*(\Omega \mid \mu) = v^*((m,M) \mid \mu) \;\; = \alpha(m,M)\, v'(M \mid \mu) + (1-\alpha(m,M))\, v'(m \mid \mu) \tag{27}$$

where v' denotes a value function for a single element.

Incorporating Dempster-Shafer probability theory in the descriptive model $f^*(\Omega,\mu)$ of a value function under uncertainty, we could model the lack of belief which cannot be modeled by Bayes' probability theory. As the result our descriptive model $f^*(\Omega,\mu)$ could resolve the Ellsberg paradox as follows.

3.3 Resolving Ellsberg paradox

Suppose an urn contains 30 balls coloured red, black or white. We know that 10 of 30 balls are red, but for the other 20 balls we know only that each of these balls is either black or white. Suppose we pick a ball from this urn, and consider four events as follows:

a. We will get 100 dollars if we pick a red ball.
b. We will get 100 dollars if we pick a black ball.
c. We will get 100 dollars if we pick a red or white ball.
d. We will get 100 dollars if we pick a black or white ball.

Many people show the preference (Ellsberg, 1961),

$$a \succ b, \;\; d \succ c. \tag{28}$$

The probability of picking up a red ball is 1/3. Let p_b and p_w be the probability of picking up a black ball and a white ball, respectively. Then

$$p_b + p_w = \frac{2}{3}. \tag{29}$$

The expected utility theory says that

$$a \succ b \;\Rightarrow\; \frac{1}{3}u(1M) > p_b u(1M) \;\Rightarrow\; p_b < \frac{1}{3} \tag{30}$$

$$d \succ c \Rightarrow \frac{2}{3}u(1M) > \frac{1}{3}u(1M) + p_w u(1M) \tag{31a}$$

$$\Rightarrow \ p_w < \frac{1}{3} \ \Rightarrow \ p_b > \frac{1}{3} \tag{31b}$$

where u denotes a von Neumann-Morgenstern utility function and 1M=100 dollars. Eqs. (30) and (31b) are obviously contradictory. This phenomenon is called the Ellsberg paradox. Therefore, the expected utility theory cannot represent the preference when the probability of each event is not known but only the basic probability for a set of events is known. This phenomenon shows that one prefers the events with known probability and is called the sure-thing principle (Kranz, *et al*., 1971).

How can we explain the preference of this Ellsberg paradox by using the descriptive model $f^*(\Omega, \mu)$ of a measurable value function under uncertainty? Let $\{R\}$ be the event of picking a red ball and $\{B,W\}$ be the set element of picking a black or white ball. Then the basic probability is written as

$$\mu(\{R\}) = \frac{1}{3}, \quad \mu(\{B,W\}) = \frac{2}{3} \tag{32}$$

In this case a set Θ containing every possible event is written as

$$\Theta = \{0, 1M\}. \tag{33}$$

Table 1: Basic probability for each event.

Alternative	Event		
	{0}	{1M}	{0, 1M}
a	2/3	1/3	0
b	1/3	0	2/3
c	0	1/3	2/3
d	1/3	2/3	0

Table 1 shows the basic probability of getting each event for each alternative. The value for each alternative is given by

$$V(a) = w'\left(\frac{2}{3}\right)v'\left(\{0\} \mid \frac{2}{3}\right) + w'\left(\frac{1}{3}\right)v'\left(\{1M\} \mid \frac{1}{3}\right) \tag{34a}$$

$$V(b) = w'\left(\frac{1}{3}\right)v'\left(\{0\} \mid \frac{1}{3}\right) + w'\left(\frac{2}{3}\right)v'\left(\{0,1M\} \mid \frac{2}{3}\right) \tag{34b}$$

$$V(c) = w'\left(\frac{1}{3}\right)v'\left(\{1M\} \mid \frac{1}{3}\right) + w'\left(\frac{2}{3}\right)v'\left(\{0,1M\} \mid \frac{2}{3}\right) \tag{34c}$$

$$V(d) = w'\left(\frac{1}{3}\right)v'\left(\{0\} \mid \frac{1}{3}\right) + w'\left(\frac{2}{3}\right)v'\left(\{1M\} \mid \frac{2}{3}\right). \tag{34d}$$

In the set Θ let x^0 and x^* be the worst outcome and the best outcome, respectively, then

$$x^0 = 0, \quad x^* = 1M \ . \tag{35}$$

Therefore, we obtain

$$v'(\{0\} \mid \mu) = 0, \quad v'(\{1M\} \mid \mu) = 1, \quad \forall \mu. \tag{36}$$

Let an index of optimism be $\alpha = \alpha(0{,}1M)$, then

$$a \succ b \;\Rightarrow\; V(a) > V(b) \tag{37a}$$

$$\Rightarrow w'\left(\frac{1}{3}\right) > w'\left(\frac{2}{3}\right) v'\left(\{0{,}1M\} \mid \frac{2}{3}\right) \tag{37b}$$

$$\Rightarrow \; w'\left(\frac{1}{3}\right) > \alpha w'\left(\frac{2}{3}\right) \tag{37c}$$

$$d \succ c \;\Rightarrow\; V(d) > V(c) \tag{38a}$$

$$\Rightarrow w'\left(\frac{2}{3}\right) > w'\left(\frac{1}{3}\right) + w'\left(\frac{2}{3}\right) v'\left(\{0{,}1M\} \mid \frac{2}{3}\right)$$

(38b)

$$\Rightarrow \; w'\left(\frac{1}{3}\right) < (1-\alpha) w'\left(\frac{2}{3}\right). \tag{38c}$$

To hold these preference relation we need to have $\alpha = \alpha(0{,}1M)$ such that

$$\frac{w'\left(\frac{1}{3}\right)}{w'\left(\frac{2}{3}\right)} > \alpha, \quad \frac{w'\left(\frac{2}{3}\right) - w'\left(\frac{1}{3}\right)}{w'\left(\frac{2}{3}\right)} > \alpha \;. \tag{39}$$

If $\alpha = \alpha(0{,}1M)$<0.5, Eq. (39) holds. This situation shows that, in general, one is pessimistic about events with unknown probability. The Ellsberg paradox is resolved by the descriptive model $f^*(\Omega, \mu)$ of a value function under uncertainty.

3.4 Potential applicability to modeling public sector decision problems

Recent increase of carbon dioxide concentration around the globe is getting serious and it is said that the resulting greenhouse effect and global warming may cause serious harm in our lives. Therefore, we need to restrict the emission of carbon dioxide and other greenhouse gas somehow. By using a value function under uncertainty shown in *3.2* we could deal with a set element like $\{d_1, d_2\}$, where

d_1 : get damage caused by unusual weather due to global warming.
d_2 : get damage caused by unusual weather which is not related with global warming.

Actually, when we get damage caused by unusual weather, we do not know whether it is due to global warming or not. Basic probability could be assigned to such a set element $\{d_1, d_2\}$. Then, we could construct a measurable value function under uncertainty for evaluating the alternative policies to decrease the emission of carbon dioxide for avoiding global warming. The measurable value function under uncertainty could also evaluate the preference of various type of decision: ordinary, pessimistic or optimistic.

4. PROSPECT THEORY UNDER UNCERTAINTY

4.1 Theory

Prospect theory (Kahneman and Tversky 1979) was proposed in order to explain people's decision making such that

(a) People's attitude to risk is loss averse.
(b) People feel that weight for very small probability is disproportionate.

We denote the prospect that yields an outcome x_j with probability $p_j, j = 1,2,\dots,n$ by Eq. (5). In prospect theory (PT), the value V for the prospect (5) is evaluated using the evaluation function

$$V = \sum_{j=1}^{n} \pi(p_j)v(x_j) \tag{40}$$

where the value function v is convex with a gentle curve in the gain domain, while it is concave and its curve is steeper in the loss domain, as shown in **Figure 1**. This shows that people, in general, are loss averse.

The weighting function π is a convex function as shown in **Figure 2**, so a small probability is weighted higher and middle or large probabilities are weighted lower. However, this weighting function is not defined near the end points 0 and 1. The dotted line in Figure 2 shows the case for the expected utility (EU) model.

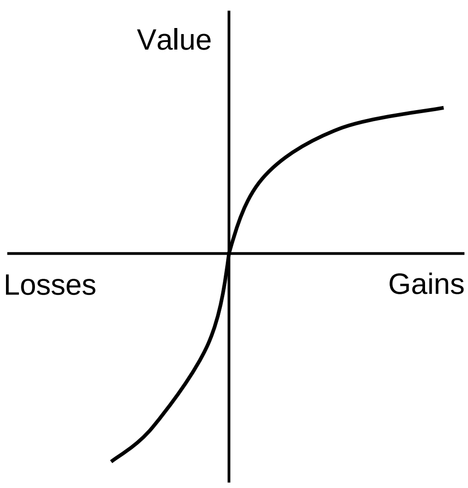

Figure 1: Value function.

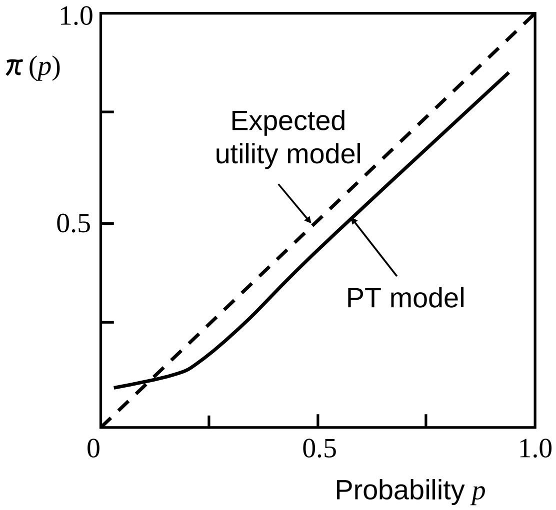

Figure 2: Weighting function used in PT.

Using PT, we are not able to deal with cases where the probability of occurrence for each event is unknown. Here we use the idea of a measurable value function under uncertainty as described in *3.2* and we develop PT under uncertainty (PTU) (Tamura and Miura, 2007).

Let the value function in PTU, based on the Dempster-Shafer basic probability, be

$$f^*(B,\mu) = \pi'(\mu)v^*(B) \tag{41}$$

as a special case of Eq. (22). Using Axiom of Dominance 1 and, by restricting a set element B to Eq. (25), Eq. (41) is reduced to

$$f^*(\Omega,\mu) = \pi'(\mu)v^*(\Omega). \tag{42}$$

However, there exist some cases for which Axiom of Dominance 1 is unsuitable. Then we introduce Axiom of Dominance 2 (Tamura and Miura 2007) which is more strict than Axiom of Dominance 1 as follows:

Axiom of Dominance 2.

Let the worst outcome be m_1 and the best outcome be M_1 in the set element B_1, and let the worst outcome be m_2 and the best outcome be M_2 in the set element B_2. Moreover, let the imaginary elements whose values are equal to the average values of B_1, B_2 be g_1, g_2, respectively. Then

$$m_1 \prec m_2, M_1 \prec M_2, g_1 \prec g_2 \Rightarrow B_1 \prec B_2 \tag{43}$$

$$m_1 \sim m_2, M_1 \sim M_2, g_1 \sim g_2 \Rightarrow B_1 \sim B_2 \tag{44}$$

where

$$v(g_1) = \frac{\sum_{i=1}^{n_1} v(a_i)}{n_1}, \quad v(g_2) = \frac{\sum_{i=1}^{n_2} v(b_i)}{n_2},$$

n_1 denotes the number of elements in the set element B_1 and n_2 denotes the number of elements in the set element B_2.

Axiom of Dominance 2 is too strict to use practically, so we try to relax it. Someone attaches importance to the best outcome and chooses an alternative, someone attaches importance to the worst outcome, and someone pays attention to the whole. We introduce the model that properly describes this situation.

Definition.

Let the elements in the set element B be $a_1, a_2, \ldots, a_n$ such that $a_i \prec a_{i+1}, \ i = 1,2,\ldots,n-1,$ the value of element $a_i, \ i = 1,2,\ldots,n$ be $v(a_i)$ and the average value of elements be

$$v(g) = \sum_{i=1}^{n} v(a_i)$$

Further, let the pessimism index decided by the question for the element $a_1 = m$ whose outcome is the worst and the element $a_n = M$ whose outcome is the best be $\alpha(m, M)$. We assume the value h of the set element B to be

$$h(B \mid \alpha) = a + be^{-c\alpha(m,M)} \qquad \text{if} \ \ v(g) \neq \frac{v(M) + v(m)}{2} \tag{45}$$

and

$$h(B \mid \alpha) = a + b\alpha(m, M) \qquad \text{if} \ \ v(g) = \frac{v(M) + v(m)}{2} \tag{46}$$

where unknown parameters a, b, c are decided by

$$h(B \mid 0) = v(M), \quad h(B \mid 0.5) = v(g), \quad h(B \mid 1) = v(m).$$

We introduce Axiom of Dominance 3 in order to evaluate values based on the above definition as follows:

Axiom of Dominance 3.

$$h(B_1 \mid \alpha) < h(B_2 \mid \alpha) \Rightarrow B_1 \prec B_2 \tag{47}$$

and

$$h(B_1 \mid \alpha) = h(B_2 \mid \alpha) \Rightarrow B_1 \sim B_2 \tag{48}$$

By using Axiom of Dominance 3, we are able to write the value function in PTU as

$$f^*(h(B \mid \alpha), \mu) = \pi'(\mu) h(B \mid \alpha). \tag{49}$$

We could properly describe the value judgment of pessimistic people and optimistic people, respectively, by using Eq. (49).

We are able to evaluate the value V of the prospect that includes the case where the probability of occurrence for each element is unknown but the basic probability of occurrence for each set element is known through the evaluation function

$$V = \sum_{j=1}^{n} \pi(\mu_j) v^*(B_j) \tag{50}$$

where π denotes the weighting function of PT and v^* denotes the value function with respect to a set element B_j.

4.2 Value Judgment of the Sense of Security for Nursing Care Robots

Japan is currently an aging society composed largely of elderly people, and the proportion of the aged in the population is increasing year by year. This causes problems because the number of people who need care is increasing every year. It is estimated that more than 4% of Japanese will need care in 2025. However, the number of nurses is less than required. In such a society, people increasingly turn to machines and tools for nursing care or welfare and some are already being put to practical use, such as wheelchairs, nursing care beds and so forth. They are becoming more popular. However, more research and development of nursing care robots is urgently required because of the diversification of people who need care.

In Tamura, *et al*., 2005 it is assumed that certain types of nursing care robot were available and tried to evaluate the sense of security that they provided. For this purpose, we used expected utility theory (von Neumann and Morgenstern, 1947), prospect theory (PT) (Kahneman and Tversky, 1979) and cumulative prospect theory (Tversky and Kahneman 1992). We conducted a survey of people who participated in nursing care activities, and compared the results. It showed that PT is the most suitable for evaluating the sense of security provided by nursing care robots among the three utility theoretic approaches above.

We also considered a more practical case in which we know the total probability for some outcomes but not the probability for each outcome (Tamura and Miura, 2007). In order to deal with such cases, we use PTU.

We now attempt to evaluate the sense of security that people feel in two different situations and compare the results. One is a situation in which all probabilities for all outcomes are known and the other is a situation in which probabilities for some outcomes are unknown but the total probability for them is known. We designate following two cases as ones of the second category where the probabilities for some outcomes are unknown.

Case 1. You do not know which will care for you, a human nurse or a nursing care robot, when you ask a nursing care center to care for you.

Case 2. You do not know what type of robot will care for you when you borrow a nursing care robot from a nursing care center or the government.

The subjects of this experiment are eleven people who live either in Ikeda City or in Minoo City in the Osaka Prefecture, Japan. Seven of them are participating in the nursing care activities and four of them have a family member who needs care. Their ages are between the twenties and fifties, and four of them are male. We assumed seven types of hypothetical nursing care robots: Robot A, Robot B, …, Robot G with different care levels, different appearance and different rental fees.

Table 2: Probability/basic probability of obtaining each outcome.

Outcome	Society 1	Society 2
No	0.35	0.25
Fa	0.35	0.35
Nu	0.30	0.30
Nu or *Ro*	0	0.40

For *Case 1* we describe outcomes and probabilities/basic probabilities in **Table 2** where *No, Fa, Nu, Ro* denote the outcomes ‘no care’, ‘care by family’, ‘care by nurse’, ‘care by robot’, respectively. In this table, Society 1 denotes the society in which nursing care robots do not exist and Society 2 denotes the society where they exist. Then the value V_1 of Society 1 and the value V_2 of Society 2, are evaluated as follows:

$$V_1 = \pi(0.35)v(No) + \pi(0.35)v(Fa) + \pi(0.30)v(Nu),$$

$$V_2 = \pi(0.25)v(No) + \pi(0.35)v(Fa) + \pi(0.40)v^*(Nu, Ro).$$

Here $v^*(Nu, Ro)$ could be described by using the pessimism index $\alpha(m, M)$ as

$$v^*(Nu, Ro) = \alpha(m, M)v(m) + (1 - \alpha(m, M))v(M)$$

where m denotes the worse outcome between *Nu* and *Ro*, and M denotes the better one between them. Using these equations, we try to evaluate each situation practically.

The results of value judgment show that the societies where any nursing care robots exist are preferred over a society where nursing care robots do not exist when every probability is known, but this preference is reversed in the situation where some probabilities are unknown. This result is consistent with the actual preference of the subject. Similarly, for all individuals, the results of evaluation are consistent with their actual preference, so we could say that PTU is a suitable model for evaluating the sense of security provided by nursing care robots in *Case 1*.

For *Case 2* the actual type of robot which cares for the subject is unknown. The value V_1 of Society 1 and the value V_2 of Society 2, are evaluated by

$$V_1 = \pi(0.35)v(No) + \pi(0.35)v(Fa) + \pi(0.30)v(Nu),$$

$$V_2 = \pi(0.25)v(No) + \pi(0.35)v(Fa) + \pi(0.30)v(Nu) + \pi(0.10)v^*(Ro).$$

where *Ro* is a set element which consists of outcomes ‘care by robot A’, ‘care by robot B’,…, ‘care by robot G’. Here $v^*(Ro)$ could be represented by using the pessimism index $\alpha(m, M)$ obtained by the question for the worst robot and the best robot as

$$v^*(Ro) = h(Ro \mid \alpha)$$

where h denotes the value function defined by Eqs. (45) and (46). The result of the value judgment indicates a preference for the situation that the type of robot is unknown over the situation that is care provided by robot C, D, E, but the reverse if care is provided by robot A, B, F, G. The value of “unknown” is lower than the average of the values of robots A, B,…, G. This result is consistent with actual preferences and every result agreed well with each individual’s preference.

We evaluated quantitatively the sense of security provided by nursing care robots in the case where probabilities for some outcomes are unknown. Furthermore, we showed that the results of evaluation correspond to the actual individuals’ preference quite well. We hope that PTU will be put into practical use in order to evaluate the value of sense of security provided by nursing care robots or something else that will be developed hereafter. We also found that people feel anxious if circumstances are not clear. It shows that publishing information is important to give people a good sense of security.

5. CONCLUDING REMARKS

Behavioral models of individual decision making have been described. In the model, outcome-dependent, non-additive probabilities are introduced as a measurable value function under risk where probability of each event occurring is postulated to be known. The effective application of this approach to the public sector is shown in modeling risks of extreme events with low probability and high outcome. The measurable value function under uncertainty is also described where the basic probability for a set of events is known but the probability of each event occurring is not. It is shown that the Ellsberg paradox is consistently resolved by using this model. The potential applicability to the global environmental problem is also noted.

By using the idea of measurable value function under uncertainty, we extended the prospect theory (PT) of Kahneman-Tversky to the prospect theory under uncertainty (PTU) and evaluated quantitatively the sense of security provided by nursing care robots in the case where probabilities for some outcomes are unknown. Furthermore, we showed that the results of this evaluation are quite consistent with the actual individuals' preference. We hope that PTU will be put to practical use in order to evaluate the sense of security provided by nursing care robots or something else that will be developed hereafter. We also found that people feel anxious if outcomes are not clear. It shows that publishing information is important to give people a good sense of security.

In this article we have described value judgment of an individual, but the value judgment of society is yet to be developed, under further research.

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